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ORIGINAL RESEARCH

Against generalized comprehension

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Abstract
Naive set theory, as practiced in the nineteenth century by Cantor, Dedekind, and Frege, consists of two basic principles: *extensionality* and *naive comprehension*. According to the latter, *every* condition determines a set. This is usually formalized by the schema $\exists y \forall x\, (x \in y \leftrightarrow A(x))$, where $A(x)$ is any condition on x in which y doesn't occur free. Some philosophers have argued that, since this restriction is in place to avoid paradox, in a logic weak enough to handle paradoxes, the restriction on $A(x)$ should be lifted in order to better capture the full generality of naive comprehension. I argue that, on the contrary, lifting the restriction doesn't do more justice to naive comprehension but is the product of a serious misunderstanding of what the principle really says.

Keywords Naive Comprehension · Paraconsistent Set-Theory · Generalized Comprehension · Existential Generalization

Naive set theory, as practiced in the nineteenth century by Cantor, Dedekind, and Frege, consists of two basic *informal* principles: *extensionality*, according to which two sets are identical if they have the same elements, and *naive comprehension*, allowing *any* condition to determine a set—the set whose elements are exactly the things of which the condition is true.[1]

[1] It isn't entirely clear that Cantor fully endorsed naive comprehension. His characterization of a set in (Cantor, 1883) as "any totality of definite elements which can be bound up into a whole by means of a *law*" (from Hallet (1984, p. 33), my emphasis) certainly suggests that much. On the other hand, a few years later in his famous letter to Dedekind (Cantor, 1967) he distinguishes between consistent and inconsistent multiplicities and states that, while every condition determines a multiplicity, only *consistent* multiplicities are sets.

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The *standard* formalization of the principle of naive comprehension is given by all the instances of the following "comprehension axiom", where $A(x)$ is any condition on x (I write $A(v_1, v_2, \dots)$ to indicate that the variables $v_1, v_2, \dots$ may occur free in A) in which y doesn't occur free:[2]

$$\exists y \forall x\, (x \in y \leftrightarrow A(x)) \tag{CA}$$

For each such $A(x)$, (CA) asserts the existence of a set y whose elements are exactly the things satisfying $A(x)$, sometimes denoted by $\{x : A(x)\}$. If y is allowed to occur free in $A(x)$, contradiction ensues. For example, let $A(x)$ be $x \notin y$. But, as Russell pointed out in a famous letter to Frege, (CA) is inconsistent even if the restriction to instances in which y isn't free in $A(x)$ is in place: letting $A(x)$ be $x \notin x$ yields a set y of all sets that aren't members of themselves, for which it follows that it must be both a member and not a member of itself.

Russell's paradox (and other so-called logical antinomies) set the community on a search for classically consistent set-theoretic axioms which turned out to be incredibly prolific. It led not only to Russell's own *Principia Mathematica*, with Whitehead, but also to the now standard Zermelo–Fraenkel set theory.

Qua classical theories, both *Principia Mathematica* and Zermelo–Fraenkel purport to "consistencize in one way or other the inconsistent", to quote Routley (later known as "Sylvan") (1977, p. 912). They do this by precluding some conditions from determining a set, that is, by dropping some—including all paradoxical—instances of the comprehension schema. There are, however, other philosophers and mathematicians—including Arruda, Brady, da Costa, Meyer, Routley, Priest, and Weber—who, since the 70s, set out to explore a more "faithful reconstruction of the original nineteenth century Cantor/Dedekind/Frege set theory" (Weber, 2012, p. 269), retaining naive comprehension but weakening the underlying logic to a paraconsistent logic, in which some classical inferences such as *ex contradictione quodlibet* fail, thus avoiding triviality.

Some of these paraconsistent set-theorists, including Brady, Routley, and Weber, at some point felt the need to go beyond the standard formalization of naive comprehension.[3] They endorsed a more general axiom-schema, which they called the "generalized comprehension axiom", consisting of *all* instances of (CA), including those

[2] Another popular way to formalize naive comprehension is by an "abstraction axiom":

$$x \in \{x : A(x)\} \leftrightarrow A(x) \tag{AA}$$

where $A(x)$ is any condition on x in which the term $\{x : A(x)\}$ doesn't occur. Brady (2006) for instance, refers to (AA) as "comprehension with terms" (although Priest (2006, ch. 2), for example, refers to (CA) itself as an abstraction principle). The difference between (AA) and (CA) is that, while (CA) merely states the existence of a set, (AA) gives it a name; otherwise, the principles are typically thought to be conceptually equivalent.

[3] See, for instance, (Routley, 1977; Brady and Routley, 1989; Brady, 2006; Weber, 2010, 2012); although see, e.g. (da Costa and Béziau, 1996; Priest, 2006, ch. 2) for paraconsistent set theories with restricted comprehension only.

in which y occurs free in $A(x)$.[4] The more general principle allows what are sometimes referred to in the literature as “circular” or “impredicative” sets such as Routley’s set $\mathcal{Z} = \{x : x \notin \mathcal{Z}\}$, a set to which something belongs just in case it doesn’t. It also allows the derivation of certain desirable principles such as Cantor’s theorem and the Axiom of Choice (cf. Routley, 1977, pp. 915–6), and it facilitates the “construction of functions, diagonal sets, and other useful objects” (Weber, 2010, p. 76).

While some of their motives are broadly pragmatic (tied to the principle’s mathematical prolificacy), the advocates of generalized comprehension offered *principled* reasons in favour of lifting the restriction, which can be found incipiently in (Routley, 1977, pp. 915–6), amplified in (Brady, 1989, p. 440; 2006, p. 176) and echoed in (Weber, 2010; 2021, ch. 1, fn 3). They argue that lifting the restrictions results in a more *faithful* formalization of the naive view of sets, as it better captures the full generality of naive comprehension. Here’s Brady:[5]

> The generalized comprehension axiom can be read as: Every formula determines the membership of some set. That is, the [generalized comprehension axiom] provides the most general form of membership determination of a set. (Brady, 1989, p. 440)

Weber (2010, p. 76) expresses a similar thought as he claims that (his version of) generalized comprehension is needed “to ensure that every predicate, even groundless ones, determines a set”. And in later work, Brady says the following:

> [...] our determination so far of what to include and exclude in the logic and its applications has been dependent on the use of natural language, and the phrase ‘a class y of all elements x’ such that $A(x, y)$, or, without the symbols, ‘a class of all elements such that...’ does produce an intuitively plausible natural lan-

[4] These authors typically formalize naive comprehension not in terms of (CA) but as an abstraction principle like the one introduced in fn 2. To obtain the generalized version, $\{x : A(x)\}$ must be allowed to occur on the right-hand side, which requires some technical maneuvering. Brady and Routley (1989, pp. 418–9), for example, add “indexed quantifiers” governed by a “reflection axiom” to the language to achieve this effect, while Brady (2006, p. 177) simply suggests that we replace (AA) with the more general principle:

$$x \in \{x : A(x, y)\} \leftrightarrow A(x, y)[\{x : A(x, y)\}/y] \tag{GAA}$$

[5] In (Brady, 2006) Brady gives two further arguments. First, the principle doesn’t lead to triviality when formulated in an appropriate paraconsistent logic. Second, generalized *propositional*: comprehension—given by *all* instances of

$$\exists p\,(p \leftrightarrow A) \tag{PC}$$

including those in which p occurs free in A—allows us to prove the existence of a ‘Liar proposition’: letting the formula A be $\neg p$ we get $\exists p\,(p \leftrightarrow \neg p)$. This, Brady maintains, is not only a desirable feature but, by allowing y to occur free in $A(x)$, (CA) would give us something analogous for sets, and equally desirable. I will not engage with these reasons any further, however, as non-triviality isn’t enough for truth, and proving the existence of a ‘Liar proposition’ or a ‘Liar set’ is hardly a positively desirable feature of a logic or theory, whether it is paraconsistent or not.

guage concept, whether y is free or not in A, that is, whatever the predicate A happens to be. (Brady, 2006, p. 176)

Given the seeming initial plausibility of having *all* instances of (CA), the restriction is perceived by those who endorse generalized comprehension—as well as by many other philosophers—as a classical attempt to avoid paradox, on a par with the other admissibility constraints usually imposed on the comprehension schema in classical systems.But then, in the context of paraconsistent set theory, where the logic is weak enough to handle contradictions, there seems to be no longer a need for any such restrictions. On the contrary, it appears that generalized comprehension better formalizes the main idea behind naive comprehension than any restricted version of the comprehension schema.

But this is simply a mistake. As I argue in what follows, generalized comprehension is not the most general form of membership determination of a set. Lifting the restriction to instances in which y isn't free in $A(x)$ does not better capture the principle of naive comprehension in its full generality but rather reflects a misunderstanding of what the principle is supposed to say.

The reason is fairly obvious. Naive comprehension doesn't just say that every condition determines a set; it says that every condition determines a particular set, namely, *the set of entities satisfying the condition*. But if we allow the very same variable that is bound by the existential quantifier in an instance of (CA) to appear free in the formula $A(x)$, the motivation for (CA) breaks down. While such instances of (CA) will indeed assert the existence of a set, there is no guarantee that the elements of that set are exactly the objects that satisfy the condition expressed by $A(x)$. So the existence of such a set is not legitimately justified by the fundamental idea behind naive comprehension. (Of course, the possibility that naive comprehension entails the existence of that set in some other way cannot be ruled out and, in fact, this will be the case in most extant paraconsistent set theories—more on this later.) In other words, if y is allowed to occur free in $A(x)$, (CA) isn't an adequate formalization of naive comprehension.[6]

An example should help to illustrate. Consider the condition $x \notin y$. For each value of y, the condition will be true exactly of the non-members of y (if any). So naive comprehension asserts that, for each such value, there is a set of things y such that $x \notin y$ —the complement of y. But this isn't what the relevant instance of (CA) for $x \notin y$ says—namely,

$$\exists y \forall x\, (x \in y \leftrightarrow x \notin y) \tag{1}$$

[6] The mistake is even more blatant in the case of (CA) (cf. fn 2 and 4). A set-theoretic abstraction principle is supposed to deliver, for each formula $A(x)$, the set of objects x of which the formula is true. If y isn't free in $A(x)$, this is exactly what (CA) does. But if y does occur free in $A(x)$ (i.e. $A(x, y)$), (CA) doesn't say that x belongs to the set of objects satisfying $A(x, y)$ just in case x satisfies the predicate $A(x, y)$, but rather just in case it satisfies $A(x, \{x : A(x, y)\})$. It is unclear how this could be justified on the basis of the naive conception of set. In particular, why should one substitute y in $A(x, y)$ with $\{x : A(x, y)\}$ and not any other term, or even at all?

Since y is also the variable of the existential quantifier, it gets 'caught' by the quantifier, losing its original value. The resulting claim is of a different form. In fact, (1) claims the existence of a very different set—Routley's set $\mathcal{Z}$, a set which is identical to its own complement. If the existence of this set is supposed to be justified directly on the basis of naive comprehension, then it should be the set of things that satisfy some given condition. But what condition? Certainly not $x \notin y$ for given values of y.[7] Without a condition holding exactly of the members of $\mathcal{Z}$, there is no instance of naive comprehension stating the existence of this set.

More generally, every instance of (CA) for a given formula $A(x)$ will entail the existence of some set, s, regardless of whether y is free in $A(x)$. However, in some cases where y does occur free in $A(x)$, the existence of s will follow even in the absence of a condition $C(x)$ that all and only the members of s satisfy, as $A(x)$ will not able to play that role. But this is exactly what the principle of naive comprehension requires. In such cases, the instance of (CA) for $A(x)$ does not provide legitimate justification for the existence of s on the basis of naive comprehension. The only *bona fide* way to establish the existence of s directly by naive comprehension is from the instance of (CA) (or some other adequate formulation of the principle) for a condition which all and only the members of s satisfy—i.e. for $C(x)$, not $A(x)$.

An analogy with a more familiar case could be helpful here. The purpose of the rule of introduction of the first-order existential quantifier is to allow us to infer the existence of an object satisfying a given condition from *any* witnessing claim, that is, any instance or statement to the effect that the condition holds of a particular object. The standard formalization of the rule is as follows: ∃

$$\text{From } A[t/v] \text{ infer } \exists v A \tag{$\exists$I}$$

where v is any (individual) variable and $A[t/v]$ is the result of replacing all free occurrences of v with the individual term t, if t is free for v in A. But we may also formalize it as:

$$\text{From } A(t) \text{ infer } \exists v A(v) \tag{$\exists$I*}$$

where $A(v)$ is the result of replacing any number of (free) occurrences of t in $A(t)$ with v. But in this case we should take care that v, the variable of the quantifier, doesn't occur free in $A(t)$. (The way in which (∃I) is formulated already precludes this from happening.) Otherwise, we could use (∃I*) to make unsound inferences. For instance, letting t be "Jack", $A(t)$ be "Jack $\neq v$", and $A(v)$ be "$v \neq v$", we could infer "$\exists v\, v \neq v$" from "Jack $\neq v$", that is, we could infer that something is different from itself from the premise that Jack is different from some given object. This

[7] One could perhaps make the case that $x \notin y$ expresses the condition "x is not a member of the very set being defined" arguing that y doesn't function here as a parameter, which can take arbitrary values, but as an anaphoric pronoun. But note that, according to the naive conception, there is no process by which sets are defined or generated by (CA). Naive sets exist 'out there' along their characterizing predicates; (CA) only expresses this fact. There are, however, other conditions in the vicinity that could in principle work, for example, "non-member of the set characterized by this very predicate". This condition may be available, but it doesn't formalize into $x \notin y$ and its formalization doesn't seem to require y to occur free in it.

example is somewhat extreme, featuring a classically contradictory conclusion. But less extreme examples also illustrate the point. Inferring that someone loves themselves (i.e. "$\exists v\, v$ loves v") from the premise that Jack loves some given person (i.e. "Jack loves v") is just as bad; perhaps there is indeed a self-loving person, but that cannot be inferred from the premise that Jack loves v.[8]

Neither of these particular inferences is licensed by the introduction rule of the existential quantifier. In both cases, the existence of an object satisfying a certain condition is inferred in the absence of a *witnessing claim*, i.e. a premise stating that the condition holds of a particular object (e.g. "Jack $\neq$ Jack" or "Jack loves Jack"), as required by the rule. The unrestricted version of ($\exists$I*) clearly isn't a faithful formalization of the rule. Analogous reasons lead us to the conclusion that generalized comprehension doesn't faithfully capture the principle of naive comprehension: the latter requires that there be a *witnessing condition* to infer the existence of a set of objects satisfying that condition, but the former doesn't always provide one.

Note as well that the introduction rule for the existential quantifier is fully general, in the sense that it licenses the inference to an existential claim from *any* instance whatsoever. But allowing the variable of the existential quantifier to occur free in $A(t)$ does nothing to better capture the full generality of the rule. Setting aside the fact that it gives rise to illegitimate instances of ($\exists$I*), other *legitimate* instances already allow us to infer an existential claim from $A(t, v)$, namely:

$$\text{From } A(t, v) \text{ infer } \exists u A(u, v) \tag{2}$$

where u is different from v and doesn't occur free in A. In an entirely parallel fashion, naive comprehension is fully general too, as it allows *any* condition whatsoever to determine a set. But allowing the variable of the existential quantifier to occur free in $A(x)$ is not a good strategy to capture the full generality of the naive principle either. Not only because it is unfaithful to the principle, but also because the restricted version of comprehension already guarantees the existence of the set $\{x : A(x, y)\}$. Renaming variables, we may infer

$$\exists z \forall x\, (x \in z \leftrightarrow A(x, y))$$

for any variable z not occurring free in $A(x, y)$.

There is an important issue that requires consideration at this point. As was later discovered by Petersen (based on a result due to Girard), generalized comprehension actually follows from standard comprehension in most logics in which paraconsistent set theories are typically formulated.[9] Since all instances of standard comprehension

[8] Introductory logic courses often expose students to ($\exists$I*)—although the rule is mostly used informally, as a sloppy formulation of ($\exists$I). It is striking, nonetheless, that students are rarely advised to be careful with free-variable occurrences. All warnings are rather exclusively focused on applications of the elimination rule of the existential quantifier or its dual, universal generalization.

[9] Restricted comprehension entails a fixed-point theorem (cf. (Girard, 1998, appendix)) from which generalized comprehension follows easily (cf. (Petersen, 2000, fn 14)) in logics where the conditional obeys detachment and conditional proof. This includes the relevant logics underlying Routley's "dialectical" set theory in (Routley, 1977) and Brady's set theory in (Brady, 2006).

are also instances of the unrestricted version, the principles turn out to be equivalent in those logics. So the argument could be made that my objection above must be misguided: if generalized comprehension is problematic, then so is every equivalent principle, including standard formulations of naive comprehension, which my argument didn't purport to undermine.[10]

My claim, however, is not that generalized comprehension is *false* but that it isn't a *faithful* formalization of the naive conception of set. And it is perfectly possible—and often the case—for a formal principle to capture (in a strict, direct sense) a certain conception without other, logically equivalent expressions doing so too. For example, while the classically valid law of excluded middle $(A \vee \neg A)$ is typically thought to express the idea that every sentence is either true or false, the paraconsistently (and classically) equivalent $\neg(A \wedge \neg A)$ isn't.[11] So the fact that the principles are equivalent in the relevant paraconsistent logics does not invalidate my argument.

For the same reasons, nor is my argument moot, as Weber (2019, p. xxi) declares the debate around generalized comprehension to be. While pragmatically it may not make a difference whether we adopt generalized comprehension as an axiom or just as a theorem, the question whether the principle adequately formalizes the naive conception of set, as Routley, Brady, and Weber maintain, is still of philosophical interest.

There is one last issue that deserves attention. Routley anticipates and responds to a potential objection to generalized comprehension that could but should not be mistaken with the point I'm trying to make here.[12] He writes:

> It may be objected that with the [generalized comprehension axiom] we lose the constructive character of set generation that the [restricted comprehension axiom] provides, with sets always eliminable through their characterising condition—whereas $\mathcal{Z}$ for example is deliberately characterised in terms of itself. But this constructive character of [naive comprehension] has always been a myth, and the paradoxes show that eliminability fails (in cases where looping occurs). Constructive generation of sets from some initially given base leads to theories of constructive sets in the vicinity of (not necessarily coinciding with) Zermelo's set theory. It does not accord with the usual or Cantorean view of sets—as given. If sets are all out there, in Aussersein, as they seem to be, then any constructive aspect vanishes. A [generalized comprehension axiom] provides a general, and legitimate, method of picking them out. (Routley, 1977, p. 915-6)

Here Routley is addressing the potential objection that, according to the naive conception, sets have a constructive character that is not captured by generalized

[10] Thanks to an anonymous reviewer for emphasizing this objection.

[11] One could argue that this is only the case because there are other, subclassical logics in which the principles aren't logically equivalent. But this is precisely the case of the restricted and the unrestricted versions of comprehension: there are paraconsistent logics—e.g. LP, the logic of paradox (cf. (Asenjo, 1966; Priest, 1979)), whose conditional does not detach—in which the restricted version does not entail its unrestricted counterpart.

[12] I am grateful to an anonymous referee for bringing this point to my attention.

comprehension. There are (at least) two different senses in which sets can be said to be constructively generated "from some initial given base" that Routley could have in mind here.

According to the iterative conception of set, embodied by Zermelo–Fraenkel set theory and others in the vicinity, sets are constructively generated in the sense that they are formed in stages, where the existence of a set can be established only after the existence of all its members has been established. So the "initial given base" from which a set is generated consists of the elements of the set. As a result, there are no self-membered or otherwise ungrounded sets—e.g. a universal set—according to the iterative conception. I completely agree with Routley that, according to the naive conception, sets are clearly not constructive in this sense, if this is what he had in mind. For some instances of restricted comprehension itself, whose ability to faithfully capture the naive principle is not in doubt here, state the existence of self-membered or otherwise ungrounded sets—e.g. letting $A(x)$ in (CA) be the condition $x = x$ expressing self-identity, *in which* y doesn't occur free, delivers a self-membered set, i.e. the set of all sets or universal set.

A perhaps more likely interpretation of what Routley has in mind when he talks about sets being constructively generated by some initial given base is that a *predicative* condition $A(x)$—that is, a condition that doesn't make any reference to the set y —must be available. (This seems obvious from Routley's claim that constructive sets are "always eliminable through their characterising condition"; the only issue with this interpretation is that Zermelo's sets aren't generated in this way.) But I agree that naive comprehension makes no such demands either, as restricted comprehension also allows some (though not all) impredicative conditions to determine sets. For example, the conditions $x \neq x$, expressing self-distinctness, and $(x \neq x \vee x \neq \varnothing)$, expressing the disjunction of self-distinctness and distinctness from the empty set, *neither of which contains* y free, both characterize the empty set $\varnothing$, but the latter mentions this very set. Another example is given by the condition $\forall z\, z \notin x$ (having no members), which determines the set $\{\varnothing\}$ and which lies in the range of the quantifier $\forall z$ occurring in its own characterizing condition.

My concern is not that generalized comprehension delivers circular, impredicative, or otherwise non-constructive sets, but that it doesn't faithfully formalize the principle of naive comprehension. Naive comprehension doesn't state that every possible set-theoretic description—e.g. "a set with infinitely many members" or "a set of things that don't belong to it"—is true of some set 'out there'. It states that every *condition* determines a set of objects *of which the condition is true*. And, as I've argued, the instances of (CA) in which y occurs free in $A(x)$ do not adequately capture this idea.

I hope I've done enough to show that, *pace* Routley, Brady, Weber, and those who sympathize with their views on generalized comprehension, the principle does not faithfully capture the general form of set determination that characterizes the naive conception of set. To believe that much is to make a conceptual mistake.

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References

Asenjo, F. G. (1966). A calculus for antinomies. *Notre Dame Journal of Formal Logic, 16*(1), 103–105.
Brady, R. (1989). The non-triviality of dialectical set theory. In G. Priest, R. Routley, & J. Norman (Eds.), *Paraconsistent logic. Essays on the inconsistent* (pp. 437–480). Munich: Philosophia Verlag.
Brady, R. (2006). *Universal logic*. Stanford: CSLI.
Brady, R., & Routley, R. (1989). The non-triviality of extensional dialectical set theory. In G. Priest, R. Routley, & J. Norman (Eds.), *Paraconsistent logic. Essays on the inconsistent* (pp. 415–436). Munich: Philosophia Verlag.
Cantor, G. (1883). Über unendliche, lineare Punktmannigfaltigkeiten, 5. *Mathematische Annalen, 21*, 545–586.
Cantor, G. (1967). Letter to Dedekind. In J. van Heijenoort (Ed.), *From Frege to Gödel: A source book in mathematical logic, 1879–1931* (pp. 113–117). Cambridge: Harvard University Press.
da Costa, N. C., & Béziau, J. (1996). Théories Paraconsistantes des Ensembles. *Logique Et Analyse, 39*, 51–67.
Girard, J.-Y. (1998). Light linear logic. *Information and Computation, 143*, 175–204.
Hallett, M. (1984). *Cantorian set theory and limitation of size*. Oxford: Clarendon Press.
Petersen, U. (2000). Logic without contraction as based on inclusion and unrestriced abstraction. *Studia Logica, 64*, 365–403.
Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic, 8*, 219–241.
Priest, G. (2006). *In contradiction*. New York: Oxford University Press.
Routley, R. (1977). Ultralogic as universal? *Relevance Logic Newsletter 2*, 51–89. Reprinted in [14], pp. 893–982.
Routley, R. (1980). *Exploring Meinong's Jungle and Beyond*. Departmental monograph number 3, Philosophy Department, RSSS, Australian National University, Canberra, Australia
Weber, Z. (2010). Transfinite numbers in paraconsistent set theory. *Review of Symbolic Logic, 3*, 71–92.
Weber, Z. (2012). Transfinite cardinals in paraconsistent set theory. *Review of Symbolic Logic, 5*, 269–293.
Weber, Z. (2019). Editor's introduction. In Z. Weber (Ed.), *Ultralogic as universal? The Sylvan Jungled* (Vol. 4, pp. 11–30). Cham: Springer.
Weber, Z. (2021). *Paradoxes and inconsistent mathematics*. Cambridge: Cambridge University Press.