Rise and Fall of the Fregean Empire

Lavinia Picollo
Assistant Professor

Seminar: *Frege's Logicism and the Neologicists*
Munich Center for Mathematical Philosophy
LMU Munich - Winter Semester 2017/18

Lower-case Greek letters $\varphi, \psi, \chi, \dots$ stand for formulae, upper-case Greek letters $\Gamma, \Delta, \dots$ for sets of formulae, lower-case Latin letter c stands for individual constants, $s, t, \dots$ for terms, $u, v, \dots$ for individual variables, upper-case Latin letter F for functor symbols, R for relation symbols, and $U, V, \dots$ for relation variables, possibly with subindexes and primes in each case.

1 First-Order Logic

1.1 Languages

Vocabulary

- Individual constants: $c_1, c_2, \dots, c_n, \dots$
- Function symbols: $f_1^1, f_2^1, \dots, f_n^1, \dots, f_1^2, f_2^2, \dots, f_n^2, \dots, f_1^m, f_2^m, \dots, f_n^m, \dots$
- Relation symbols: $=, R_1^1, R_2^1, \dots, R_n^1, \dots, R_1^2, R_2^2, \dots, R_n^2, \dots, R_1^m, R_2^m, \dots, R_n^m, \dots$
- Individual variables: $x_1, x_2, \dots, x_n, \dots$
- Logical symbols: $\neg, \rightarrow, \forall, =$
- Auxiliary symbols: (,), ,

K is a set containing individual constants, function symbols, and relation symbols. K must always contain at least one relation symbol. We define the first-order language $\mathscr{L}_K$ as follows.

Terms of $\mathscr{L}_K$

- Individual constants in K are terms of $\mathscr{L}_K$.
- Individual variables are terms of $\mathscr{L}_K$.
- If $t_1, \dots, t_n$ are terms of $\mathscr{L}_K$ and $f_i^n \in K$, then $f_i^n(t_1, \dots, t_n)$ is a term of $\mathscr{L}_K$.
- Only strings of symbols obtained applying the other clauses in the definition a finite number of times are terms of $\mathscr{L}_K$.

Closed terms are those in which no variables occur.

Well-formed Formulae (wff) of $\mathscr{L}_K$

- If $t_1, \ldots, t_n$ are terms of $\mathscr{L}_K$ and $R_i^n \in K$, then $R_i^n(t_1, \ldots, t_n)$ is an atomic wff of $\mathscr{L}_K$.
- If φ is a wff of $\mathscr{L}_K$, then $\neg\varphi$ is a wff of $\mathscr{L}_K$.
- If φ and ψ are wff of $\mathscr{L}_K$, then $(\varphi \to \psi)$ is a wff of $\mathscr{L}_K$.
- If φ is a wff of $\mathscr{L}_K$, then $\forall v\varphi$ is a wff of $\mathscr{L}_K$.
- Only strings of symbols obtained applying clauses 1-4 a finite number of times are wff of $\mathscr{L}_K$.

We omit brackets, subscripts, and superscripts when not needed. We write $x, y, z, \ldots$ for $x_1, x_2, x_3, \ldots$. If φ is a wff of $\mathscr{L}_K$, we write $\varphi \in \mathscr{L}_K$.

Sentences are wff in which every individual variable v that occurs in them occurs under the scope of $\forall v$. That is, there are no free (unbound) variables in sentences.

t is *free in φ for v* iff no free occurrence of v in φ is in the scope of a quantifier followed by a variable that occurs in t. $\varphi[t/v]$ is the result of *substituting* all free occurrences of v in φ with t, if t is free in φ for v.

Shortcuts

- $s \neq t := \neg s = t$;
- $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$;
- $\varphi \vee \psi := \neg\varphi \to \psi$;
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$;
- $\exists v\varphi := \neg\forall v\neg\varphi$;
- $\exists! v\varphi(v) := \exists v(\varphi(v) \wedge \forall u(\varphi(u) \to u = v))$.

1.2 Deductive systems

Axioms of $\mathsf{D1}$*:*[1]

A1 $\varphi \to (\psi \to \varphi)$

A2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

A3 $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

A4 $\forall v\varphi \to \varphi[t/v]$, if t is free for v in φ.

Rules of inference of $\mathsf{D1}$*:*

$$\frac{\varphi, \varphi \to \psi}{\psi}\ (\textit{Modus Ponens}) \qquad\qquad \frac{\varphi \to \psi}{\varphi \to \forall v\psi}\ (\textit{Generalisation})$$

if v is not free in φ or in a premise.

φ is a *proof-theoretic consequence* of Γ in $\mathsf{D1}$ ($\Gamma \vdash_{\mathsf{D1}} \varphi$) iff there is a deduction of φ from Γ

[1] As is usual, we take the axioms to be the universal closure of the instances of the following axiom-schemata.

in $\mathsf{D1}$, that is, there is a sequence of formulae s.t. each member is either an axiom of $\mathsf{D1}$, or a member of Γ, or was obtained by one of the rules of inference of $\mathsf{D1}$ from previous members of the sequence, and the last member is φ. If Γ is empty, φ is a *logical theorem* ($\vdash_{\mathsf{D1}} \varphi$).

The axioms and rules of inference of $\mathsf{D1}^=$ are those for $\mathsf{D1}$ plus the following:

=1 $\forall x\, x = x$

=2 $\forall x \forall y (x = y \to (\varphi(x) \to \varphi(y)))$, if y is free for x in φ.

A *theory* in $\mathsf{D1}$ ($\mathsf{D1}^=$) is a set of formulae of $\mathscr{L}_K$, closed under proof-theoretic consequence in $\mathsf{D1}$ ($\mathsf{D1}^=$) (i.e. if φ is a proof-theoretic consequence of some members of the theory, φ is also a member of the theory).

1.3 Semantics

A *model* of $\mathscr{L}_K$ is a structure $\mathcal{M} = \langle d, \mathcal{I} \rangle$, in which d is a non-empty set, the *domain* of the model, and $\mathcal{I}$ is an interpretation function, s.t.:

- if $c \in K$, $\mathcal{I}(c) \in d$;
- if $f_i^n \in K$, $\mathcal{I}(f_i^n)$ is a function from d^n to d;
- if $R_i^n \in K$, $\mathcal{I}(R_i^n) \subseteq d^n$.

A *variable assignment* σ in $\mathcal{M}$ is any function form the set of individual variables to d. It can be extended to an *assignment* to all terms of $\mathscr{L}_K$ as follows:

- if $c \in K$, $\sigma(c) = \mathcal{I}(c)$;
- if $f_i^n \in K$ and $t_1, \ldots, t_n$ are terms of $\mathscr{L}_K$, $\sigma(f_i^n(t_1, \ldots, t_n)) = \mathcal{I}(f_i^n)(\sigma(t_1), \ldots, \sigma(t_n))$.

Satisfaction by an assignment in a model

φ is satisfied in $\mathcal{M}$ by σ ($\mathcal{M}, \sigma \vDash \varphi$) iff

- $\varphi := (s = t)$ and $\sigma(s) = \sigma(t)$; or
- $\varphi := R_i^n t_1, \ldots, t_n$ and $\langle \sigma(t_1), \ldots, \sigma(t_n) \rangle \in \mathcal{I}(R_i^n)$; or
- $\varphi := \neg\psi$ and $\mathcal{M}, \sigma \nvDash \psi$; or
- $\varphi := (\psi \to \chi)$ and $\mathcal{M}, \sigma \nvDash \psi$ or $\mathcal{M}, \sigma \vDash \chi$; or
- $\varphi := \forall v \psi$ and $\mathcal{M}, \sigma' \vDash \psi$ for every assignment σ' that differs from σ at most in the value assigned to v.

φ is *true in* $\mathcal{M}$ ($\mathcal{M} \vDash \varphi$) iff, for every assignment σ in $\mathcal{M}$, $\mathcal{M}, \sigma \vDash \varphi$. We write $\mathcal{M} \vDash \Gamma$ to indicate that $\mathcal{M} \vDash \varphi$ for every $\varphi \in \Gamma$.

φ is a *semantic consequence* of Γ ($\Gamma \vDash \varphi$) iff, for every model $\mathcal{M}$ and assignment σ in $\mathcal{M}$, if $\mathcal{M}, \sigma \vDash \Gamma$, then $\mathcal{M}, \sigma \vDash \varphi$. If Γ is empty, φ is a *logical truth* ($\vDash \varphi$).

Exercise 1. *Show that (=1) and (=2) are logical truths (the latter by induction on the complexity of φ).*

1.4 Metatheoretic Results

Theorem 1 (Soundness.)**.** *If* $\Gamma \vdash_{\mathsf{D1}^=} \varphi$*, then* $\Gamma \vDash \varphi$*.*

Theorem 2 (Completeness.)**.** *If* $\Gamma \vDash \varphi$*, then* $\Gamma \vdash_{\mathsf{D1}^=} \varphi$*.*

Theorem 3 (Compactness.)**.** *If* $\Gamma \vDash \varphi$*, then there is a finite* $\Delta \subseteq \Gamma$ *s.t.* $\Delta \vDash \varphi$*.*

Theorem 4 (Downward Löwenheim-Skolem.)**.** *If* $\mathcal{M} \vDash \Gamma$ *and* d *is infinite, then there is a model* $\mathcal{M}' = \langle d', \mathcal{I}' \rangle$ *s.t.* $\mathcal{M}' \vDash \Gamma$ *and* d' *has cardinality* $\aleph_0$ *(i.e. the cardinality of the set of natural numbers).*

Theorem 5 (Upward Löwenheim-Skolem.)**.** *If* $\mathcal{M} \vDash \Gamma$ *and* d *is infinite, then for each infinite cardinal number* κ *there is a model* $\mathcal{M}_\kappa = \langle d_\kappa, \mathcal{I}_\kappa \rangle$ *s.t.* $\mathcal{M}_\kappa \vDash \Gamma$ *and* d_κ *has cardinality* κ*.*

1.5 Arithmetic

$\mathrm{A} := \{0, \mathrm{S}, +, \times, =\}$, where 0 is an individual constant, S a unary function symbol, and $+$ and $\times$ binary function symbols. $\mathscr{L}_\mathrm{A}$ is the language of first-order arithmetic.

Peano Arithmetic, $\mathsf{PA} \subseteq \mathscr{L}_\mathrm{A}$ in $\mathsf{D1}^=$ consists of the following axioms:

PA1 $\forall x\ \mathrm{S}x \neq 0$

PA2 $\forall x \forall y (\mathrm{S}x = \mathrm{S}y \to x = y)$

PA3 $\forall x\ x + 0 = x$

PA4 $\forall x \forall y\ x + \mathrm{S}y = \mathrm{S}(x + y)$

PA5 $\forall x\ x \times 0 = 0$

PA6 $\forall x \forall y\ x \times \mathrm{S}y = x \times y + x$

Induction schema. $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(\mathrm{S}x)) \to \forall x \varphi$

$\mathcal{N} := \langle \mathbb{N}, \mathcal{I}_\mathbb{N} \rangle$ is the model of $\mathscr{L}_\mathrm{A}$ s.t.

- $\mathbb{N} = \{0, 1, 2, \dots\}$;
- $\mathcal{I}_\mathbb{N}(0) = 0$;
- $\mathcal{I}_\mathbb{N}(S) = \{\langle x, y \rangle \in \mathbb{N}^2 : y = x + 1\}$ (the successor function over $\mathbb{N}$);
- $\mathcal{I}_\mathbb{N}(+) = \{\langle x, y, z \rangle \in \mathbb{N}^3 : z = x + y\}$ (the addition function over $\mathbb{N}^2$);
- $\mathcal{I}_\mathbb{N}(\times) = \{\langle x, y, z \rangle \in \mathbb{N}^3 : z = x \times y\}$ (the multiplication function over $\mathbb{N}^2$).

$\mathcal{N}$ is the *standard* or *intended* model of $\mathscr{L}_\mathrm{A}$. $\mathscr{L}_\mathrm{A}$ contains a name

$$\underbrace{\mathrm{S} \dots \mathrm{S}}_{n \text{ times}}\ 0$$

for each $n \in \mathbb{N}$, which we note $\bar{n}$, and call the *numeral* of n.

A *non-standard* model of $\mathscr{L}_\mathrm{A}$ is a model of $\mathscr{L}_\mathrm{A}$ that is not isomorphic with $\mathcal{N}$.[2]

[2] Two models $\mathcal{M} = \langle d, \mathcal{I} \rangle$ and $\mathcal{M}' = \langle d', \mathcal{I}' \rangle$ of $\mathscr{L}_K$ are isomorphic iff d and d' have the same cardinality and there is a bijective function $j : d \to d'$ s.t.

- if $c \in K$, $j(\mathcal{I}(c)) = \mathcal{I}'(c)$;

Exercise 2. *Show that $\mathcal{N} \vDash \mathsf{PA}$.*

Theorem 6 (Gödel's First Incompleteness)**.** *For every effectively axiomatisable theory Th*[3] *formulated in an extension of $\mathscr{L}_{\mathrm{A}}$, that is consistent and contains PA1-PA6, there is a $\varphi \in \mathscr{L}_{\mathrm{A}}$ s.t. $\mathcal{N} \vDash \varphi$ but $Th \nvDash \varphi$.*

Corollary 7. PA *has non-standard, countable and uncountable, models.*

Proof. Let $\varphi \in \mathscr{L}_{\mathrm{A}}$ be s.t. $\mathcal{N} \vDash \varphi$ but $\mathsf{PA} \nvDash \varphi$. Then, $\mathsf{PA} \cup \{\neg\varphi\}$ is consistent. Since PA has only infinite models, so does $\mathsf{PA} \cup \{\neg\varphi\}$. By the Löwenheim-Skolem results, this theory has countable and uncountable models. Both are, a fortiori, non-standard models of PA. □

2 Second-Order Logic

2.1 Languages

In second-order languages the vocabulary, class of terms, formulae, and sentences extend their first-order counterparts as follows.

New Vocabulary

- Functor symbols:[4] $F_1, F_2, \ldots, F_n, \ldots$
- Relation variables: $X_1^1, X_2^1, \ldots, X_n^1, \ldots, X_1^2, X_2^2, \ldots, X_n^2, \ldots, X_1^m, X_2^m, \ldots, X_n^m, \ldots$

We write X_i and R_i instead of X_i^1 and R_i^1, for perspicuity. For the same reason, we often write $X, Y, Z, \ldots$ for $X_1, X_2, X_3, \ldots$.

K need no longer contain at least one relation symbol. Figuring out why is left as an **exercise**.

Terms of $\mathscr{L}_K^2$

The set of terms of $\mathscr{L}_K^2$ extends that of $\mathscr{L}_K$ with the following:

- If $F, R_i \in K$, then $F(R_i)$ is a term of $\mathscr{L}_K^2$.
- If $F \in K$, then $F(X_i)$ is a term of $\mathscr{L}_K^2$.

Well-formed Formulae (wff) of $\mathscr{L}_K^2$

The set of wff of $\mathscr{L}_K^2$ extends that of $\mathscr{L}_K$ with the following:

- If $t_1, \ldots, t_n$ are terms of $\mathscr{L}_K^2$, then $X_i^n(t_1, \ldots, t_n)$ is an atomic wff of $\mathscr{L}_K^2$.
- If φ is a wff of $\mathscr{L}_K^2$, then $\forall V \varphi$ is a wff of $\mathscr{L}_K^2$.

Sentences are wff in which every individual variable v that occurs in them occurs under the scope of $\forall v$, and every relation variable V that occurs in them occurs under the scope of $\forall V$.

- if $f_i^n \in K$ and $d_1, \ldots, d_n \in d$, $j(\mathcal{I}(f_i^n)(d_1, \ldots, d_n)) = \mathcal{I}'(f_i^n)(j(d_1), \ldots, j(d_n))$;
- if $R_i^n \in K$ and $d_1, \ldots, d_n \in d$, $\langle d_1, \ldots, d_n\rangle \in \mathcal{I}(R_i^n)$ iff $\langle j(d_1), \ldots, j(d_n)\rangle \in \mathcal{I}'(R_i^n)$.

Note that if two models are isomorphic, they make true the same formulae.

[3] There is an algorithm to find out whether a formula is an axiom of the theory.

[4] For simplicity, we just introduce monadic functor symbols, that is, that should be followed syntactically just by one monadic relation variable. Functor symbols rarely occur in second-order languages, but play a central role in Frege's project.

X_i^n is *free in* φ *for* X_j^n iff no free occurrence of X_j^n in φ is in the scope of a quantifier of the form $\forall X_i^n$.

Let T_i, T_j be relation variables or symbols of the same arity. $\varphi[T_i/T_j]$ is the result of *substituting* all (free) occurrences of T_j in φ with T_i, if T_i is free in φ for T_j.

New shortcuts

- $s = t := \forall X(Xs \leftrightarrow Xt)$;
- $\exists V\varphi := \neg\forall V\neg\varphi$.

2.2 Deductive system

$\mathsf{D2}$ extends $\mathsf{D1}$ with the following:

Axioms of $\mathsf{D2}$*:*

A5 $\forall X_i^n\varphi \to \varphi[T/X_i^n]$, if T is an n-ary relation variable free for X_i^n in φ or an n-ary relation symbol.

Comprehension schema. $\exists X^n\forall x_1 \ldots \forall x_n(X^n x_1 \ldots x_n \leftrightarrow \varphi(x_1, \ldots, x_n))$, if X^n is not free in φ.

Rules of inference of $\mathsf{D2}$*:*

$$\frac{\varphi \to \psi}{\varphi \to \forall V\psi}\ (\textit{Second-order generalisation})$$

if V is not free in φ or in a premise.

Proof-theoretic consequence, *logical theoremhood*, and *theory* in $\mathsf{D2}$ are defined as in $\mathsf{D1}$.

Exercise 3. *Show that (=1) and (=2) are logical theorems in* $\mathsf{D2}$*.*

2.3 Semantics

Standard Semantics

A *standard model* $\mathcal{M} = \langle d, \mathcal{I}\rangle$ of $\mathscr{L}_K^2$ is just a model of $\mathscr{L}_K$, plus an interpretation of the functor symbols as follows:

- $\mathcal{I}(F)$ is a function from $\wp(d)$ to d.[5]

An *assignment* σ in a model $\mathcal{M} = \langle d, \mathcal{I}\rangle$ of $\mathscr{L}_K^2$ extends an assignment for the first-order case to relation variables and new terms as follows:

- $\sigma(X_i^n) \subseteq d^n$;
- if $F, R_i \in K$, then $\sigma(F(R_i)) = \mathcal{I}(F)(\mathcal{I}(R_i))$;
- if $F \in K$, then $\sigma(F(X_i)) = \mathcal{I}(F)(\sigma(X_i))$.

Satisfaction by an assignment in a model

[5] Given a set s, $\wp(s)$ is the power set of s, the set of subsets of s, i.e. $\wp(s) = \{s' : s' \subseteq s\}$.

Satisfaction in a standard model of $\mathscr{L}_K$ extends to formulae of $\mathscr{L}_K^2$ as follows: $\mathcal{M}, \sigma \vDash \varphi$ iff

- $\varphi := X_i^n t_1, \dots, t_n$ and $\langle \sigma(t_1), \dots, \sigma(t_n) \rangle \in \sigma(X_i^n)$; or
- $\varphi := \forall V \psi$ and $\mathcal{M}, \sigma' \vDash \psi$ for every assignment σ' that differs from σ at most in the value assigned to V.

Truth in a standard model, *standard semantic consequence*, and *standard logical truth* for $\mathscr{L}_K^2$ are defined as for the first-order case.

Henkin Semantics

A *Henkin model* of $\mathscr{L}_K^2$ is a structure $\mathcal{M} = \langle d, D, \mathcal{I} \rangle$, in which d is a non-empty set, the *first-order domain* of the model, D is a set of subsets $D(n)$ of $\wp(d_n)$ (i.e. $D = \{D(n) : n \in \mathbb{N}\}$, and $D(n) \subseteq \wp(d_n)$), the class of *second-order domains* of the model, and $\mathcal{I}$ is an interpretation function just as in the first-order case, plus the following clause for functors:

- $\mathcal{I}(F)$ is a function from $D(1)$ to d.

An *assignment* σ in a Henkin model $\mathcal{M} = \langle d, D, \mathcal{I} \rangle$ of $\mathscr{L}_K^2$ extends an assignment for the first-order case as in the standard case, except in case of relation variables:

- $\sigma(X_i^n) \in D(n)$.

Satisfaction by an assignment in a Henkin model, *truth in a Henkin model*, *Henkin-semantic consequence* ($\Gamma \vDash_{\mathrm{H}} \varphi$), and *Henkin-logical truth* ($\vDash_{\mathrm{H}} \varphi$) for $\mathscr{L}_K^2$ are defined just like in standard semantics.

Exercise 4. *Show that there are Henkin models in which some instances of comprehension are not true.*

Henkin models in which the comprehension schema comes out true are called *faithful models*. Henkin models in which each $D(n)$ coincides with $\wp(d_n)$ are called *full models*.

Exercise 5. *Show that every standard model is equivalent (makes the same formulae true) to a full Henkin model.*

Exercise 6. *Show that if φ is Henkin-valid, then φ is standardly valid.*

2.4 Arithmetic

$\mathscr{L}_{\mathrm{A}}^2$ is the language of second-order arithmetic.

Second-order Peano Arithmetic, $\mathsf{Z}_2 \subseteq \mathscr{L}_{\mathrm{A}}^2$ in $\mathsf{D2}$, consists of the axioms PA1-6 plus the following:

Induction axiom. $\forall X(X0 \land \forall x(Xx \to X\mathrm{S}x) \to \forall x Xx)$

Exercise 7. *Show that $\mathcal{N} \vDash \mathsf{Z}_2$.*

Theorem 8 (Categoricity)**.** *If $\mathcal{M} \vDash \mathsf{Z}_2$, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Exercise 8. *Show that, for every $\varphi \in \mathscr{L}_{\mathrm{A}}^2$, $\mathsf{Z}_2 \vDash \varphi$ or $\mathsf{Z}_2 \vDash \neg\varphi$.*

2.5 Metatheoretical Results

For Standard Semantics

Theorem 9 (Soundness.)**.** *If* $\Gamma \vdash_{\mathsf{D2}} \varphi$*, then* $\Gamma \vDash \varphi$*.*

Theorem 10 (Inherent incompleteness.)**.** *For every effectively axiomatisable extension* D *of* $\mathsf{D2}$*, there exist* Γ, φ *s.t.* $\Gamma \vDash \varphi$ *but* $\Gamma \nvdash_D \varphi$*.*

Theorem 11 (Failure of compactness.)**.** *For some infinite* Γ *and some* φ *it is the case that* $\Gamma \vDash \varphi$ *but then there is no finite* $\Delta \subseteq \Gamma$ *s.t.* $\Delta \vDash \varphi$*.*

Theorem 12 (Failure of downward Löwenheim-Skolem.)**.** *For some* Γ *and some infinite model* $\mathcal{M} \vDash \Gamma$ *there is no model* $\mathcal{M}' = \langle d', \mathcal{I}' \rangle$ *s.t.* $\mathcal{M}' \vDash \Gamma$ *and* d' *has cardinality* $\aleph_0$*.*

Theorem 13 (Failure of upward Löwenheim-Skolem.)**.** *For some* Γ *and some infinite model* $\mathcal{M} \vDash \Gamma$ *it is not the case that, for each infinite cardinal number* κ*, there is a model* $\mathcal{M}_\kappa = \langle d_\kappa, \mathcal{I}_\kappa \rangle$ *s.t.* $\mathcal{M}_\kappa \vDash \varphi$ *and* d_κ *has cardinality* κ*.*

For (Faithful) Henkin Semantics

We take Henkin Logical consequence to be defined only in terms of faithful Henkin models in this section.

Theorem 14 (Soundness.)**.** *If* $\Gamma \vdash_{\mathsf{D2}} \varphi$*, then* $\Gamma \vDash_{\mathrm{H}} \varphi$*.*

Theorem 15 (Completeness.)**.** *If* $\Gamma \vDash_{\mathrm{H}} \varphi$*, then* $\Gamma \vdash_{\mathsf{D2}} \varphi$*.*

Theorem 16 (Compactness.)**.** *If* $\Gamma \vDash_{\mathrm{H}} \varphi$*, then there is a finite* $\Delta \subseteq \Gamma$ *s.t.* $\Delta \vDash_{\mathrm{H}} \varphi$*.*

Theorem 17 (Downward Löwenheim-Skolem.)**.** *If* $\mathcal{M} \vDash_{\mathrm{H}} \Gamma$ *and* d *is infinite, then there is a model* $\mathcal{M}' = \langle d', D', \mathcal{I}' \rangle$ *s.t.* $\mathcal{M}' \vDash_{\mathrm{H}} \Gamma$ *and* d' *has cardinality* $\aleph_0$*.*

Theorem 18 (Upward Löwenheim-Skolem.)**.** *If* $\mathcal{M} \vDash_{\mathrm{H}} \Gamma$ *and* d *is infinite, then for each infinite cardinal number* κ *there is a model* $\mathcal{M}_\kappa = \langle d_\kappa, D_\kappa, \mathcal{I}_\kappa \rangle$ *s.t.* $\mathcal{M}_\kappa \vDash_{\mathrm{H}} \Gamma$ *and* d_κ *has cardinality* κ*.*

3 Definitions

Let $c, f_i^n, R_i^n, F \in K$, let $Th \subseteq \mathscr{L}_K$ ($\mathscr{L}_K^2$) be a theory in $\mathsf{D1}^=$ ($\mathsf{D2}$), and assume c, f_i^n, R_i^n, F do not occur in any (non-logical) axiom of Th. Also, let $v_1, \ldots, v_n, u$ be distinct variables. The only free variables allowed to occur in an expression are explicitly indicated in brackets.

3.1 Explicit

Individual Constants

An *explicit definition* of c in Th is a formula of the form

$$c = t,$$

where t is a closed term of $\mathscr{L}_K$ ($\mathscr{L}_K^2$) in which c doesn't occur.

Alternatively, an *explicit definition* of c can be given by a formula of the form

$$\forall v (c = v \leftrightarrow \varphi(v)), \tag{1}$$

provided that $\varphi \in \mathscr{L}_K$ ($\mathscr{L}_K^2$) is s.t. v is the only free variable, c doesn't occur in it, and φ is true of one and only one object, i.e. $\exists! v \varphi(v)$.

Exercise 9. *Indicate how a formula of the form (1) could fail to define* c *if* $\exists! v \varphi(v)$ *fails to be true.*

Function Symbols

An *explicit definition* of f_i^n in *Th* is a formula of the form

$$f_i^n(v_1, \dots, v_n) = t(v_1, \dots, v_n),$$

where t is a term of $\mathscr{L}_K$ ($\mathscr{L}_K^2$) in which f_i^n doesn't occur.

Alternatively, an *explicit definition* of f_i^n can be given by a formula of the form

$$\forall v_1 \dots \forall v_n \forall u (f_i^n(v_1, \dots, v_n) = u \leftrightarrow \varphi(v_1, \dots, v_n, u)), \tag{2}$$

where $\varphi \in \mathscr{L}_K$ ($\mathscr{L}_K^2$) is s.t. $v_1, \dots, v_n, u$ are the only free variables, f_i^n doesn't occur in it, and it expresses a function, i.e. for every $v_1, \dots, v_n$ there is a unique u s.t. φ is true. Formally,

$$\forall v_1 \dots \forall v_n \exists! u \varphi(v_1, \dots, v_n, u). \tag{3}$$

Exercise 10. *Indicate how a formula of the form (2) could fail to define f_i^n if (3) is not true.*

Relation Symbols

An *explicit definition* of R_i^n in *Th* is a formula of the form

$$\forall v_1 \dots \forall v_n (R_i^n(v_1, \dots, v_n) \leftrightarrow \varphi(v_1, \dots, v_n)), \tag{4}$$

where $\varphi \in \mathscr{L}_K$ ($\mathscr{L}_K^2$) is s.t. $v_1, \dots, v_n$ are the only free variables and R_i^n doesn't occur in it.

Exercise 11. *Indicate how a formula of the form (4) could fail to define R_i^n if φ contains free variables other than $v_1, \dots, v_n$.*

Functor Symbols

An *explicit definition* of F in *Th* is a formula of the form

$$F(X_i) = t(X_i),$$

where t is a term of $\mathscr{L}_K^2$ in which F doesn't occur.

Alternatively, an *explicit definition* of F can be given by a formula of the form

$$\forall X_i \forall v (F(X_i) = v \leftrightarrow \varphi(X_i, v)), \tag{5}$$

where $\varphi \in \mathscr{L}_K^2$ is s.t. X_i and v are the only free variables, F doesn't occur in it, and φ expresses a function, i.e. for every X_i there is a unique v s.t. φ is true. Formally,

$$\forall X_i \exists! v \varphi(X_i, v). \tag{6}$$

Exercise 12. *Indicate how a formula of the form (5) could fail to define F if (6) is not true.*

3.2 Recursive

Let $\mathscr{L}_K$ ($\mathscr{L}_K^2$) extend $\mathscr{L}_\mathrm{A}$ ($\mathscr{L}_\mathrm{A}^2$).

Function Symbols

A *recursive definition* of f_i^n in *Th* is given by the universal closure of the conjunction of

- $f_i^n(v_1, \ldots, v_{n-1}, 0) = s(v_1, \ldots, v_{n-1})$;
- $f_i^n(v_1, \ldots, v_{n-1}, \mathrm{S}v_n) = t(v_1, \ldots, v_n)$;

where s, t are terms of $\in \mathscr{L}_K$ ($\mathscr{L}_K^2$) s.t. f_i^n doesn't occur in s, and it only features in t in subterms of the form $f_i^n(v_1, \ldots, v_n)$.

Alternatively, a *recursive definition* of f_i^n can be given by the universal closure of the conjunction of the following:

- $f_i^n(v_1, \ldots, v_{n-1}, 0) = u \leftrightarrow \varphi(v_1, \ldots, v_{n-1}, u)$;
- $f_i^n(v_1, \ldots, v_{n-1}, \mathrm{S}v_n) = u \leftrightarrow \psi(v_1, \ldots, v_n, u)$;

where $\varphi, \psi \in \mathscr{L}_K$ ($\mathscr{L}_K^2$) are s.t. f_i^n doesn't occur in φ, and it only features in ψ in subterms of the form $f_i^n(v_1, \ldots, v_n)$.

Exercise 13. *Show that a recursive definition of f_i^n of the latter kind is logically equivalent to*

$$\forall v_1 \ldots \forall v_n \forall u (f_i^n(v_1, \ldots, v_n) = u \leftrightarrow (v_n = 0 \wedge \varphi(v_1, \ldots, v_{n-1}, u)) \vee \exists v (v_n = \mathrm{S}v \wedge \psi(v_1, \ldots, v_n, u)))$$

Relation Symbols

A *recursive definition* of R_i^n in *Th* is given by the universal closure of the conjunction of

- $R_i^n(v_1, \ldots, v_{n-1}, 0) \leftrightarrow \varphi(v_1, \ldots, v_{n-1})$;
- $R_i^n(v_1, \ldots, v_{n-1}, \mathrm{S}v_n) \leftrightarrow \psi(v_1, \ldots, v_n)$;

where $\varphi, \psi \in \mathscr{L}_K$ ($\mathscr{L}_K^2$) are s.t. R_i^n doesn't occur in φ, and the only occurrences of R_i^n in ψ are of the form $R_i^n(v_1, \ldots, v_n)$.

Exercise 14. *Show that a recursive definition of R_i^n is logically equivalent to*

$$\forall v_1 \ldots \forall v_n (R_i^n(v_1, \ldots, v_n) \leftrightarrow (v_n = 0 \wedge \varphi(v_1, \ldots, v_{n-1})) \vee \exists u (v_n = \mathrm{S}u \wedge \psi(v_1, \ldots, v_{n-1}, u)))$$

3.3 Metatheoretic Results

Theorem 19 (Eliminability.)**.** *If δ is an explicit definition of an expression ϵ in $Th \subseteq \mathscr{L}_K$ ($\mathscr{L}_K^2$), then for every wff φ of $\mathscr{L}_K$ ($\mathscr{L}_K^2$), there is a wff φ' of $\mathscr{L}_K$ ($\mathscr{L}_K^2$) in which ϵ doesn't occur, s.t. $Th, \delta \vdash_{\mathsf{D1=(D2)}} \varphi \leftrightarrow \varphi'$.*

Corollary 20 (Non-creativity.)**.** *If δ is an explicit definition of an expression ϵ in $Th \subseteq \mathscr{L}_K$ ($\mathscr{L}_K^2$), then there is no wff φ of $\mathscr{L}_K$ ($\mathscr{L}_K^2$) s.t. $Th, \delta \vdash_{\mathsf{D1=(D2)}} \varphi$ but $Th \nvdash_{\mathsf{D1=(D2)}} \varphi$.*

Theorem 21. *If δ is a recursive definition of an expression ϵ in a theory $Th \subseteq \mathscr{L}_K^2$ extending Z_2, there is a $\delta' \in \mathscr{L}_K^2$ s.t. $Th \vdash_{\mathsf{D2}} \delta \leftrightarrow \delta'$ and δ' is an explicit definition of ϵ in Th.*

4 Fregean Foundations

4.1 Frege's Logic

Let $\mathrm{F} := \mathrm{A} \cup \{\mathrm{N}, \S, \#\}$, where N is a unary relation symbol, and § and # are functor symbols for unary relation symbols and variables. $\mathscr{L}_\mathrm{F}^2$ is the language of Frege's Logic.

Frege's Logic, FL, extends $\mathsf{D2}$ with the following axiom:

Basic Law V (BLV). $\forall X \forall Y(\S X = \S Y \leftrightarrow X \equiv Y)$

where $X \equiv Y := \forall x(Xx \leftrightarrow Yx)$ ($X \equiv Y$ iff X and Y hold of the same objects), plus definitions for $\#, 0, \mathrm{S}, +, \times$, and N, as indicated in what follows. The intended reading of BLV is that concepts that hold of the same objects have the same extension/form the same class.

4.2 Natural Numbers and Hume's Principle

$X \approx Y := \exists X^2(\forall x(Xx \to \exists! y(Yy \wedge X^2 xy)) \wedge \forall y(Yy \to \exists! x(Xx \wedge X^2 xy)))$ (there is a bijection between X and Y)

We can define 'number of', the functor $\#$, in FL as follows:

$\forall X(\# X =_{def} \S(\exists Y(x = \S Y \wedge X \approx Y)))$[6]

Exercise 15. *Show that $\approx$ is an equivalence relation, that is:*

1. $\forall X X \approx X$,
2. $\forall X \forall Y(X \approx Y \to Y \approx X)$,
3. $\forall X \forall Y \forall Z(X \approx Y \wedge Y \approx Z \to X \approx Z)$.

Then one could define each numeral as follows:

- $0 =_{def} \#(x \neq x)$
- $\bar{1}$ $(= \mathrm{S}0) =_{def} \#(x = 0)$
- $\bar{2}$ $(= \mathrm{SS}0) =_{def} \#(x = 0 \vee x = 1)$
- ...

But this is not a definition of S, only a list of its instances. As such, it won't help us derive the axioms of PA or Z_2. Thus, more work needs to be done.

$\mathrm{Pred}(x, y) := \exists X(Xx \wedge y = \# X \wedge x = \#(Xz \wedge z \neq x))$

$\mathrm{Her}(X) := \forall x \forall y(\mathrm{Pred}(x, y) \to (Xx \to Xy))$

$\mathrm{Ans}(x, y) := \forall X(Xx \wedge \mathrm{Her}(X) \to Xy)$

Exercise 16. *Show that*

1. $\mathrm{Pred}(x, y) \to \mathrm{Ans}(x, y)$
2. $\mathrm{Pred}(x, y) \wedge \mathrm{Pred}(y, z) \to \mathrm{Ans}(x, z)$
3. $\mathrm{Ans}(x, y) \wedge \mathrm{Ans}(y, z) \to \mathrm{Ans}(x, z)$
4. $\neg \mathrm{Pred}(x, x)$
5. $\mathrm{Ans}(x, x)$

[6] Given that formation rules for terms with functors only allow for functors to syntactically apply to relation variables or symbols, this is just an abbreviation of

$$\forall X \forall y(\# X = y \leftrightarrow \exists Z(y = \S(Z) \wedge \forall x(Zx \leftrightarrow \exists Y(x = \S Y \wedge X \approx Y)))).$$

The same goes for the definitions of particular numerals coming next.

Definitinons of arithmetical symbols in $\mathscr{L}_{\mathrm{A}}^2$

- $\forall x(\mathrm{N}x \leftrightarrow_{def} \mathrm{Ans}(0, x))$
- $\forall x \forall y(\mathrm{S}x = y \leftrightarrow_{def} \mathrm{N}x \wedge \mathrm{Pred}(x, y))$
- $\forall x \forall y \forall z(x + y = z \leftrightarrow_{def} \mathrm{N}x \wedge \mathrm{N}y \wedge (y = 0 \wedge z = x) \vee \exists w(w = \mathrm{S}y) \wedge z = \mathrm{S}(x + y))$[7]
- $\forall x \forall y \forall z(x \times y = z \leftrightarrow_{def} \mathrm{N}x \wedge \mathrm{N}y \wedge (y = 0 \wedge z = 0) \vee \exists w(w = \mathrm{S}y \wedge z = x \times y + x))$[8]

Theorem 22. *The following is provable in* FL*:*

Hume's Principle (HP). $\forall X \forall Y(\#X = \#Y \leftrightarrow X \approx Y)$

Proof. Left as an **exercise**. (HP follows easily from the definition of #.) □

The intended reading of HP is that the number of X is the same as that of Y just in case there is a bijection between these two concepts.

4.3 Frege's Theorem

Exercise 17. *Show that the following follow from the definitions of* $\mathrm{N}, 0, \mathrm{S}, +, \times$ *and HP:*

1. $\mathrm{N}0$
2. $\forall x \forall y \forall X \forall Y(Xx \wedge Yy \wedge X \approx Y \rightarrow (Xz \wedge z \neq x) \approx (Yz \wedge z \neq y))$
3. $\forall x \forall y \forall X \forall Y(Xx \wedge Yy \wedge (Xz \wedge z \neq x) \approx (Yz \wedge z \neq y) \rightarrow X \approx Y)$
4. $\forall x \forall y(\mathrm{N}x \wedge \mathrm{N}y \wedge \mathrm{S}x = \mathrm{S}y \rightarrow x = y)$
5. $\forall x \forall y(\mathrm{N}x \wedge \mathrm{Pred}(x, y) \rightarrow \mathrm{N}y)$
6. $\forall x(\mathrm{N}x \rightarrow \exists! y(y = \mathrm{S}x \wedge \mathrm{N}y))$
7. $\forall X(\forall x(\mathrm{N}x \rightarrow (Xx \rightarrow X\mathrm{S}x)) \rightarrow \forall x \forall y(\mathrm{Pred}(x, y) \rightarrow (\mathrm{N}x \wedge Xx \rightarrow \mathrm{N}y \wedge Xy)))$

Theorem 23 (Frege)**.** *The axioms of* Z_2 *(relativised to* $\mathrm{N}v$*) are provable in* FL*.*

Proof. We show they all follow from the definitions of $\mathrm{N}, 0$, S, $+, \times$ and HP, without making use of BLV.

1.	$\mathrm{N}x \wedge \mathrm{S}x = 0$	assumption
2.	$\mathrm{N}x \wedge \mathrm{S}x = \#(x \neq x)$	Def. 0, 1
3.	$\mathrm{Pred}(x, \#(x \neq x))$	Def. S, 2
4.	$\exists X(Xx \wedge \#(x \neq x) = \#X \wedge x = \#(Xz \wedge z \neq x))$	Def. Pred, 3

[7] This is just long for

$$x + 0 =_{def} x;$$
$$x + \mathrm{S}y =_{def} \mathrm{S}(x + y).$$

[8] This is just long for

$$x \times 0 =_{def} 0;$$
$$x \times \mathrm{S}y =_{def} x \times y + x.$$

5. $x \neq x$ HP, 4
6. $\forall x(\mathrm{N}x \to \mathrm{S}x \neq 0)$ 1-5

1. $\mathrm{N}x \wedge \mathrm{N}y \wedge \mathrm{S}x = \mathrm{S}y$ assumption
2. $\mathrm{Pred}(x, \mathrm{S}y) \wedge \mathrm{Pred}(y, \mathrm{S}x)$ Def. S, 1
3. $\exists X(Xx \wedge \mathrm{S}y = \#X \wedge x = \#(Xz \wedge z \neq x))$ Def. Pred, 2
4. $\exists X(Xx \wedge \mathrm{Pred}(y, \#X) \wedge x = \#(Xz \wedge z \neq x))$ Def. S, 3
5. $\exists X(Xx \wedge \exists Y(Yy \wedge \#X = \#Y \wedge y = \#(Yz \wedge z \neq y)) \wedge x = \#(Xz \wedge z \neq x))$ Def. Pred, 4
6. $\exists X \exists Y(Xx \wedge Yy \wedge X \approx Y \wedge y = \#(Yz \wedge z \neq y) \wedge x = \#(Xz \wedge z \neq x))$ HP, 5
7. $\exists X \exists Y((Xz \wedge z \neq x) \approx (Yz \wedge z \neq y) \wedge y = \#(Yz \wedge z \neq y) \wedge x = \#(Xz \wedge z \neq x))$ Ex. 17.2, 6
8. $\exists X \exists Y(\#(Xz \wedge z \neq x) = \#(Yz \wedge z \neq y) \wedge y = \#(Yz \wedge z \neq y) \wedge x = \#(Xz \wedge z \neq x))$ HP, 7
9. $x = y$ $=^2$, 8
10. $\forall x \forall y(\mathrm{N}x \wedge \mathrm{N}y \to (\mathrm{S}x = \mathrm{S}y \to x = y))$ 1-9

Axioms PA3-PA6 follow trivially. They are left as **exercises**. For the induction axiom we have the following:

1. $X0 \wedge \forall x(\mathrm{N}x \to (Xx \to X\mathrm{S}x))$ assumption
2. $\mathrm{N}x$ assumption
3. $\mathrm{Ans}(0, x)$ Def. N, 2
4. $\forall X(X0 \wedge \mathrm{Her}(X) \to Xx)$ Def. Ans, 3
5. $\mathrm{N}0 \wedge X0 \wedge \mathrm{Her}(\mathrm{N}x \wedge Xx) \to (\mathrm{N}x \to Xx)$ 4
6. $\forall x \forall y(\mathrm{Pred}(x, y) \to (\mathrm{N}x \wedge Xx \to \mathrm{N}y \wedge Xy))$ 17.7, 1
7. $\mathrm{Her}(\mathrm{N}x \wedge Xx)$ Def. Her, 6
8. $\mathrm{N}x \to Xx$ Ex. 17.1, 1, 5, 7
9. $\forall x(\mathrm{N}x \to Xx)$ 2-8

$\square$

4.4 Russell's Paradox

$\mathrm{C}(x) \;:=\; \exists X\; x = \S X$

$x \in y \;:= \exists X(y = \S X \wedge Xx)$

$\mathrm{C}(x)$'s intended to mean that x is a class or extension, whereas $x \in y$ is to indicate that x is a member of the class y.

Lemma 24. *The following follow in* FL*:*

Comprehension schema for classes/extensions. $\forall X \exists x \forall y(y \in x \leftrightarrow Xy)$

Extensionality principle. $\forall x \forall y(\mathrm{C}(x) \wedge \mathrm{C}(y) \to (\forall z(z \in x \leftrightarrow z \in y) \to x = y))$

Proof. Left as an **exercise**. $\square$

Theorem 25 (Russell)**.** FL *is inconsistent (and, thus, trivial).*

Proof.

1. $\exists x \forall y (y \in x \leftrightarrow y \notin y)$ — Comprehension for classes
2. $\forall y (y \in x \leftrightarrow y \notin y)$ — assumption
3. $x \in x \leftrightarrow x \notin x$ — 2
4. $\perp$ — 1, 2-3

$\square$

References

[1] Shapiro, S. *Foundations without Foundationalism: A Case for Second-Order Logic.* Oxford University Press, New York, 1991.

[2] Zalta, E. N. Frege's Theorem and Foundations for Arithmetic. In *The Stanford Encyclopedia of Philosophy*, E. N. Zalta, Ed., Summer 2017 ed. Metaphysics Research Lab, Stanford University, 2017.