Gödel's First Incompleteness Theorem
Unit 2: So many theories of arithmetic

Lavinia Picollo

1 Baby Arithmetic

Definition (L_B)**.** *L_B is just like L_A but without variables and quantifiers.*

L_B doesn't have open terms; all its terms are L_A's closed terms.

Exercise 1. *Give a (recursive) definition of the class of wffs of L_B.*

Definition (Baby Arithmetic)**.** BA *is the theory whose language is L_B, whose logic is classical propositional logic plus standard identity rules, and whose non-logical axioms are every instance of the following schemas:*

Schema 1. $\mathsf{0} \neq \mathsf{S}\overline{\mathsf{n}}$

Schema 2. $\mathsf{S}\overline{\mathsf{m}} = \mathsf{S}\overline{\mathsf{n}} \to \overline{\mathsf{m}} = \overline{\mathsf{n}}$

Schema 3. $\overline{\mathsf{n}} + \mathsf{0} = \overline{\mathsf{n}}$

Schema 4. $\overline{\mathsf{m}} + \mathsf{S}\overline{\mathsf{n}} = \mathsf{S}(\overline{\mathsf{m}} + \overline{\mathsf{n}})$

Schema 5. $\overline{\mathsf{n}} \times \mathsf{0} = \mathsf{0}$

Schema 6. $\overline{\mathsf{m}} \times \mathsf{S}\overline{\mathsf{n}} = (\overline{\mathsf{m}} \times \overline{\mathsf{n}}) + \overline{\mathsf{m}}$

Proposition 1. $\mathsf{BA} \vdash \mathsf{0} \neq \overline{\mathsf{1}}$

Proposition 2. $\mathsf{BA} \vdash \overline{\mathsf{4}} \neq \overline{\mathsf{2}}$

Theorem 6. *If $m \neq n$, then* $\mathsf{BA} \vdash \overline{\mathsf{m}} \neq \overline{\mathsf{n}}$.

Exercise 2. *Prove Theorem 6.*

Theorem 7. BA *can prove any true equation of the form $\overline{\mathsf{m}} + \overline{\mathsf{n}} = \overline{\mathsf{k}}$ and $\overline{\mathsf{m}} \times \overline{\mathsf{n}} = \overline{\mathsf{k}}$.*

Exercise 3. *Prove Theorem 7.*

Theorem 8. BA *can prove any true equation of the form $\tau = \overline{\mathsf{n}}$, where τ is a term of L_B.*

Exercise 4. *Prove Theorem 8.*

Theorem 9. *Let σ and τ be terms of L_B. If $\sigma = \tau$ is true, then $\mathsf{BA} \vdash \sigma = \tau$; if $\sigma = \tau$ is false, then $\mathsf{BA} \vdash \sigma \neq \tau$.*

Proof. Assume $\sigma = \tau$ is true. Then, there is a natural number n s.t. both $\sigma = \overline{\mathsf{n}}$ and $\tau = \overline{\mathsf{n}}$ are true. By Theorem 8, we know that $\mathsf{BA} \vdash \sigma = \overline{\mathsf{n}}$ and $\mathsf{BA} \vdash \tau = \overline{\mathsf{n}}$. By the laws of identity, $\mathsf{BA} \vdash \sigma = \tau$.

Assume now that $\sigma = \tau$ is false. Thus, there are natural numbers m, n s.t. $m \neq n$ and both $\sigma = \overline{\mathsf{m}}$ and $\tau = \overline{\mathsf{n}}$ are true. By Theorems 6 and 8, we have that $\mathsf{BA} \vdash \overline{\mathsf{m}} \neq \overline{\mathsf{n}}$, $\mathsf{BA} \vdash \sigma = \overline{\mathsf{m}}$, and $\mathsf{BA} \vdash \tau = \overline{\mathsf{n}}$. By the laws of identity, $\mathsf{BA} \vdash \sigma \neq \tau$. $\square$

Theorem 10. BA *is negation complete.*

Exercise 5. *Prove Theorem 10 by induction on the complexity of the wff.*

2 Robinson Arithmetic

2.1 The theory

Definition (Robinson Arithmetic)**.** Q *is the first-order theory formulated in L_A whose non-logical axioms are the following:*

Axiom 1. $\forall \mathsf{x}\,(0 \neq \mathsf{Sx})$

Axiom 2. $\forall \mathsf{x} \forall \mathsf{y}\,(\mathsf{Sx} = \mathsf{Sy} \rightarrow \mathsf{x} = \mathsf{y})$

Axiom 3. $\forall \mathsf{x}\,(\mathsf{x} \neq 0 \rightarrow \exists \mathsf{y}\,(\mathsf{x} = \mathsf{Sy}))$

Axiom 4. $\forall \mathsf{x}\,(\mathsf{x} + 0 = \mathsf{x})$

Axiom 5. $\forall \mathsf{x} \forall \mathsf{y}\,(\mathsf{x} + \mathsf{Sy} = \mathsf{S}(\mathsf{x} + \mathsf{y}))$

Axiom 6. $\forall \mathsf{x}\,(\mathsf{x} \times 0 = 0)$

Axiom 7. $\forall \mathsf{x} \forall \mathsf{y}\,(\mathsf{x} \times \mathsf{Sy} = \mathsf{x} \times \mathsf{y} + \mathsf{x})$

Theorem 12. $\mathsf{Q} \nvdash \forall \mathsf{x}\,(0 + \mathsf{x} = \mathsf{x})$

Proof. Let N^* be just like the standard model of L_A, N, except the domain contains two new objects, a and b, s.t. $S^*a = a$ and $S^*b = b$, and $a +^* n = a$, $b +^* n = b$, $n +^* a = b$, and $n +^* b = a$. (The functions assigned to S, $+$, and $\times$ behave exactly like in N w.r.t. the standard natural numbers). One can easily check that Axioms 1-7 are true in N^* but $\forall \mathsf{x}\,(0 + \mathsf{x} = \mathsf{x})$ is not. $\square$

Theorem 13. Q *is negation incomplete.*

2.2 Capturing $\leq$

Definition (capturability/representability)**.** *Let T be formulated in L_A and φ be a wff of L_A.*

- $\varphi(\mathsf{x})$ *captures the numerical property P in T iff, for every natural number n, if n has the property P, $T \vdash \varphi(\overline{\mathsf{n}})$ and, if n doesn't have property P, then $T \vdash \neg\varphi(\overline{\mathsf{n}})$.*
- $\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{m})$ *captures the numerical relation R in T iff, for every n-tuple of natural numbers $n_1, \ldots, n_m$, if $n_1, \ldots, n_m$ are related via R, $T \vdash \varphi(\overline{\mathsf{n}_1}, \ldots, \overline{\mathsf{n}_\mathsf{m}})$ and, if they aren't, then $T \vdash \neg\varphi(\overline{\mathsf{n}_1}, \ldots, \overline{\mathsf{n}_\mathsf{m}})$.*

Exercise 6. *Give a numerical property P, a wff $\varphi(\mathsf{x})$ of L_A, and a theory T formulated in L_A s.t. P is expressed in L_A by $\varphi(\mathsf{x})$ but not captured by this wff in T.*

Exercise 7. *Give a numerical property P, a wff $\varphi(\mathsf{x})$ of L_A, and a theory T formulated in L_A s.t. P is not expressed in L_A by $\varphi(\mathsf{x})$ but captured by this wff in T.*

Theorem 14. *The less-than-or-equal-to relation is captured in Q by $\exists \mathsf{v}\,(\mathsf{v}+\mathsf{x}=\mathsf{y})$.*

Proof. Suppose $m \leq n$. Then, there's a k s.t. $k+m=n$. Since trivially $\mathsf{BA} \subseteq \mathsf{Q}$, by Theorem 9 we have that $\mathsf{Q} \vdash \overline{\mathsf{k}}+\overline{\mathsf{m}}=\overline{\mathsf{n}}$ and, by existential weakening, that $\mathsf{Q} \vdash \exists \mathsf{v}\,(\mathsf{v}+\overline{\mathsf{m}}=\overline{\mathsf{n}})$.

Assume now that $m>n$ (so $m \neq 0$ and $m-n \neq 0$). We reason (schematically) in Q:

1. $\exists \mathsf{v}\,(\mathsf{v}+\overline{\mathsf{m}}=\overline{\mathsf{n}})$ — assumption
2. $\mathsf{v}+\overline{\mathsf{m}}=\overline{\mathsf{n}}$ — assumption
3. $\mathsf{v}+\underbrace{\mathsf{S}\ldots\mathsf{S}}_{m\text{ times}}\mathsf{0}=\underbrace{\mathsf{S}\ldots\mathsf{S}}_{m\text{ times}}(\mathsf{v}+\mathsf{0})$ — Axiom 5
4. $\underbrace{\mathsf{S}\ldots\mathsf{S}}_{m\text{ times}}(\mathsf{v}+\mathsf{0})=\underbrace{\mathsf{S}\ldots\mathsf{S}}_{n\text{ times}}\mathsf{0}$ — 2,3
5. $\underbrace{\mathsf{S}\ldots\mathsf{S}}_{m-n\text{ times}}(\mathsf{v}+\mathsf{0})=\mathsf{0}$ — Axiom 2, 4
6. $\bot$ — Axiom 1
7. $\bot$ — 1, 2-6 (E$\exists$)
8. $\neg\exists \mathsf{v}\,(\mathsf{v}+\overline{\mathsf{m}}=\overline{\mathsf{n}})$ — 1-7

□

Notation. We will use the following abbreviations in L_A:

- $\mathsf{x} \leq \mathsf{y} :\leftrightarrow\ \exists \mathsf{v}\,(\mathsf{v}+\mathsf{x}=\mathsf{y})$
- $\forall \mathsf{v} \leq \tau\, \varphi :\leftrightarrow\ \forall \mathsf{v}\,(\mathsf{v} \leq \tau \rightarrow \varphi)$ (if τ is a term not containing v)
- $\exists \mathsf{v} \leq \tau\, \varphi :\leftrightarrow\ \exists \mathsf{v}\,(\mathsf{v} \leq \tau \wedge \varphi)$ (if τ is a term not containing v)

Definition (bounded quantifiers)**.** *(An occurrence of) a quantifier $\forall\mathsf{v}$ or $\exists\mathsf{v}$ in a wff is bounded iff it's immediately followed by $(\mathsf{v} \leq \tau \rightarrow \varphi)$ or $(\mathsf{v} \leq \tau \wedge \varphi)$, resp., for some term τ not containing v and some wff φ.*

Proposition 3.

1. $\mathsf{Q} \vdash \forall \mathsf{x}\,(\mathsf{x} \leq \overline{\mathsf{n}} \vee \overline{\mathsf{n}} \leq \mathsf{x})$ *(the proof is very tedious)*
2. $\mathsf{Q} \vdash \forall \mathsf{x}\,(\mathsf{x}=\mathsf{0} \vee \mathsf{x}=\overline{\mathsf{1}} \vee \cdots \vee \mathsf{x}=\overline{\mathsf{n}} \leftrightarrow \mathsf{x} \leq \overline{\mathsf{n}})$
3. *If $\mathsf{Q} \vdash \varphi(\mathsf{0}) \wedge \varphi(\overline{\mathsf{1}}) \wedge \cdots \wedge \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash \forall \mathsf{x} \leq \overline{\mathsf{n}}\, \varphi(\mathsf{x})$.*
4. *If $\mathsf{Q} \vdash \varphi(\mathsf{0}) \vee \varphi(\overline{\mathsf{1}}) \vee \cdots \vee \varphi(\overline{\mathsf{n}})$, then $\mathsf{Q} \vdash \exists \mathsf{x} \leq \overline{\mathsf{n}}\, \varphi(\mathsf{x})$.*

Exercise 8. *Prove items 2-4 of Proposition 3.*

2.3 Arithmetical complexity

Definition (Δ_0)**.** *An L_A wff is Δ_0 just in case all its quantifiers (except those occurring in subformulas of the form $\tau \leq \sigma$) are bounded.*

Δ_0 formulas express decidable properties/relations/functions.

Theorem 16. Q *correctly decides all* Δ_0 *sentences.*

Definition (Σ_1, Π_1)**.** *An* L_A *wff is* Σ_1 *iff it is logically equivalent to a* Δ_0 *wff preceded at most by one string of existential quantifiers. An* L_A *wff is* Π_1 *iff it is logically equivalent to a* Δ_0 *wff preceded at most by one string of universal quantifiers.*

Proposition 4.

1. *The negation of a* Δ_0 *formula is* Δ_0.
2. Δ_0 *formulas are also* Σ_1 *and* Π_1.
3. *The negation of a* Σ_1 *formula is* Π_1 *and vice versa.*
4. *If* φ *and* ψ *are* Σ_1, *then so are* $\varphi \wedge \psi$ *and* $\varphi \vee \psi$.
5. *If* φ *is* Σ_1, *then so are* $\forall \mathsf{v} \leq \tau\, \varphi$ *and* $\exists \mathsf{v} \leq \tau\, \varphi$ *(if* v *doesn't occur in* τ*)*

Proof. 5. If φ is Σ_1, then it is logically equivalent to a wff of the form $\exists \vec{\mathsf{u}}\, \psi$ where ψ is Δ_0. Thus, $\forall \mathsf{v} \leq \tau\, \varphi$ and $\exists \mathsf{v} \leq \tau\, \varphi$ are logically equivalent to $\forall \mathsf{v} \leq \tau\, \exists \vec{\mathsf{u}}\, \psi$ and $\exists \mathsf{v} \leq \tau\, \exists \vec{\mathsf{u}}\, \psi$, resp. In turn, $\forall \mathsf{v} \leq \tau\, \exists \vec{\mathsf{u}}\, \psi$ is logically equivalent to $\exists \vec{\mathsf{w}} \forall \mathsf{v} \leq \tau \exists \vec{\mathsf{u}} \leq \vec{\mathsf{w}}\, \psi$ and $\exists \mathsf{v} \leq \tau\, \exists \vec{\mathsf{u}}\, \psi$ directly to $\exists \vec{\mathsf{u}} \exists \mathsf{v} \leq \tau\ \psi$ (check these facts using natural deduction!). Since ψ is Δ_0, $\forall \mathsf{v} \leq \tau\, \varphi$ and $\exists \mathsf{v} \leq \tau\, \varphi$ are Σ_1. $\square$

Exercise 9. *Prove items 1-4 of Proposition 4.*

Definition (Σ_{n+1}, Π_{n+1})**.** *An* L_A *wff is* Σ_{n+1}, *with* $n > 0$, *iff it is logically equivalent to a* Π_n *wff preceded at most by one string of existential quantifiers. An* L_A *wff is* Π_{n+1}, *with* $n > 0$, *iff it is logically equivalent to a* Σ_n *wff preceded at most by one string of universal quantifiers.*

Theorem 17 (Σ_1-completeness)**.** Q *proves all true* Σ_1 *sentences.*

Proof. Let φ be a true Σ_1 sentence. Then, φ is logically equivalent to a sentence of the form $\exists \mathsf{v_1} \ldots \mathsf{v_n}\, \psi$, where all quantifiers in ψ are bounded. Thus, $\exists \mathsf{v_1} \ldots \mathsf{v_n}\, \psi$ is also true, so there exist natural numbers $m_1, \ldots, m_n$ s.t. $\psi[\overline{\mathsf{m_1}}/\mathsf{v_1}] \ldots [\overline{\mathsf{m_n}}/\mathsf{v_n}]$ is true as well. By Theorem 16, Q proves $\psi[\overline{\mathsf{m_1}}/\mathsf{v_1}] \ldots [\overline{\mathsf{m_n}}/\mathsf{v_n}]$ and, by existential weakening, it also proves $\exists \mathsf{v_1} \ldots \mathsf{v_n}\, \psi$. By the Completeness Theorem, $\mathsf{Q} \vdash \varphi$. $\square$

Theorem 18. *If* T *is a consistent theory formulated in* L_A *which includes* Q, *then every* Π_1 *sentence that it proves is true.*

3 Peano Arithmetic

Definition (Peano Arithmetic)**.** PA *is the first-order theory formulated in* L_A *whose non-logical axioms are the following:*

Axiom 1. $\forall \mathsf{x}\, (0 \neq \mathsf{Sx})$

Axiom 2. $\forall \mathsf{x} \forall \mathsf{y}\, (\mathsf{Sx} = \mathsf{Sy} \rightarrow \mathsf{x} = \mathsf{y})$

Axiom 3. $\forall \mathsf{x}\, (\mathsf{x} + 0 = \mathsf{x})$

Axiom 4. $\forall \mathsf{x} \forall \mathsf{y}\, (\mathsf{x} + \mathsf{Sy} = \mathsf{S}(\mathsf{x} + \mathsf{y}))$

Axiom 5. $\forall \mathsf{x}\, (\mathsf{x} \times 0 = 0)$

Axiom 6. $\forall \mathsf{x} \forall \mathsf{y}\, (\mathsf{x} \times \mathsf{Sy} = \mathsf{x} \times \mathsf{y} + \mathsf{x})$

Induction Schema. $[\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(Sx))] \rightarrow \forall x\,\varphi(x)$

Proposition 5. *The following are theorems of* PA*:*

1. $\forall x\,(x \neq 0 \rightarrow \exists y\,(x = Sy))$
2. $\forall x\,(0 + x = x)$
3. $\forall x\,(x \neq Sx)$

Proof. 1. We can prove $0 \neq 0 \rightarrow \exists y\,(0 = Sy)$ in PA, as the antecedent is provably false. We reason in PA:

1. $x \neq 0 \rightarrow \exists y\,(x = Sy)$ assumption
2. $Sx \neq 0$ assumption
3. $Sx = Sx$
4. $\exists y\,(Sx = Sy)$ 3
5. $Sx \neq 0 \rightarrow \exists y\,(Sx = Sy)$ 2-4
6. $[x \neq 0 \rightarrow \exists y\,(x = Sy)] \rightarrow [Sx \neq 0 \rightarrow \exists y\,(Sx = Sy)]$ 1-5
7. $\forall x\,([x \neq 0 \rightarrow \exists y\,(x = Sy)] \rightarrow [Sx \neq 0 \rightarrow \exists y\,(Sx = Sy)])$ 6

Thus, by letting φ in the Induction Schema be $x \neq 0 \rightarrow \exists y\,(x = Sy)$, we have our result. □

Exercise 10. *Prove items 2 and 3 of Proposition 5.*

Proposition 6. PA *has countable non-standard models.*

Proof. Let L_A^c extend L_A with a new individual constant c and let PA^c be the theory extending PA with the following axioms: $c \neq 0$, $c \neq \overline{1}$, $c \neq \overline{2}$, ... – i.e. $c \neq \overline{n}$, for each natural number n. Trivially, every finite set of axioms of PA^c has a (standard) model – just let c denote $n+1$, where n is the maximum number s.t. $c \neq \overline{n}$ is in the set. By Compactness, PA^c has a model too and, by Löwenheim-Skolem, it has a countable model. Since $PA \subseteq PA^c$, this countable non-standard model is also a model of PA. □

4 Presburger Arithmetic

Definition (Presburger Arithmetic)**.** *Presburger Arithmetic is the first-order theory formulated in L_A whose non-logical axioms are the following:*

Axiom 1. $\forall x\,(0 \neq Sx)$

Axiom 2. $\forall x \forall y\,(Sx = Sy \rightarrow x = y)$

Axiom 3. $\forall x\,(x + 0 = x)$

Axiom 4. $\forall x \forall y\,(x + Sy = S(x + y))$

Induction Schema. $[\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(Sx))] \rightarrow \forall x\,\varphi(x)$

Theorem. *Presburger Arithmetic is negation complete.*