Gödel's First Incompleteness Theorem

Unit 3: Primitive Recursion

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We said an effectively decidable numerical property/relation/function is one for which there is an *algorithm*, a finite set of mechanical instructions, that allows us decide, for each number/numbers, if it has the property/they stand in the relation/their image. So, what is an algorithm? More precisely, what counts as a mechanical instruction and what counts as an acceptable way of combining mechanical instructions?

In 1936 Alan Turing proposed that an algorithm be identified with a Turing machine, i.e. a program (in the computer science sense) given in the language he puts forward (the first programming language!), consisting of certain basic instructions and ways of combining them. Accordingly, he proposed that the decidable numerical properties/relations/functions are those that can be decided by an algorithm in his sense, for which there is a Turing machine that computes them. This class coincides with the class of recursive numerical properties/relations/functions, of which the primitive recursive (p.r.) functions are a proper (but large) subset.

1 Primitive recursive functions

The p.r. functions consist of a set of basic functions and all the other functions that can be obtained from them following two combination modes: composition and primitive recursion.

1.1 The initial functions

Definition (initial functions)**.** *The initial functions are the following:*

- *The successor function, S: given a natural number, it returns its successor.*
- *The zero function, Z: given a natural number, it returns* 0.
- *The identity (projection) functions, I^k_i (where i, k are natural numbers and $1 \leq i \leq k$): given a k-tuple of natural numbers, they return the i-th.*

1.2 Composition and primitive recursion

Let $\vec{v}$ abbreviate the sequence of variables $v_1, \ldots, v_n$ (possibly of length 0).

Definition (composition)**.** *Let $g(\vec{y})$ be a function or a constant (in which case $\vec{y}$ has length 0) and $h(\vec{x}, u, \vec{z})$ a function. If $f(\vec{x}, \vec{y}, \vec{z}) := h(\vec{x}, g(\vec{y}), \vec{z})$, then f is defined by composition by substituting g into h.*

The function O that maps every natural number to 1 can be defined by composing S with Z as $O(x) := S(Z(x))$. And the function P that adds two to each number can be defined by composing S with itself as $P(x) := S(S(x))$.

Definition (primitive recursion). *Let $g(\vec{x})$ be a function or a constant and $h(\vec{x}, y, z)$ be a function. If*

- $f(\vec{x}, 0) := g(\vec{x})$, *and*
- $f(\vec{x}, Sy) := h(\vec{x}, y, f(\vec{x}, y))$

then f is defined from g and h by primitive recursion.

Addition can be defined from I_1^1 and S (composed with I_3^3) by primitive recursion as follows:

- $x + 0 := I_1^1(x)$
- $x + Sy := S(x + y) = S(I_3^3(x, y, x + y))$

Multiplication can be defined from Z and $+$ (composed with I_3^3 and I_1^3) by primitive recursion as follows:

- $x \times 0 := Z(x)$
- $x \times Sy := x \times y + x = I_3^3(x, y, x \times y) + I_1^3(x, y, x \times y)$

The function D that doubles each natural number can be defined by composing $\times$ with P as $D(x) := P(x) \times x$.

The factorial function can be defined from 1 and $\times$ composed with (I_2^2 and) S (composed with I_1^2) by primitive recursion as follows:

- $0! := 1$
- $Sx! := x! \times Sx = I_2^2(x, x!) \times S(I_1^2(x, x!))$

Exponentiation can be defined from O and $\times$ by primitive recursion as follows:

- $x^0 := O(x)$
- $x^{Sy} := x^y \times x$

1.3 Primitive recursive functions

Definition (primitive recursive functions). *The p.r. functions are as follows:*

- *All initial functions are p.r.*
- *If f can be defined from the p.r. functions or constants g and h by composition, then f is p.r.*
- *If f can be defined from the p.r. functions or constants g and h by primitive recursion, then f is p.r.*
- *Nothing else is a p.r. function.*

Definition (full definition). *A full definition for the p.r. function f is a specification of a (finite) sequence of functions $f_0, f_1, f_2, \ldots, f_k$ where each f_j is either an initial function or is defined from previous functions in the sequence by composition or recursion, and $f_k = f$.*

Exercise 1. *Show that the following functions are p.r.:*

1. *The predecessor function P:* $P(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - 1 & \text{if } x > 0 \end{cases}$
2. *The subtraction function $\dot{-}$:* $x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$
3. *The sign function sg:* $sg(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$
4. *The inverted sign function $\overline{sg}$:* $\overline{sg}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$
5. *Absolute difference $|x - y|$: given two natural numbers, x and y, it returns the distance between them.*

Proposition 1. *Let $f(x)$ be a p.r. function. Then the following are also p.r.:*

1. *The summation function, Σ:* $\Sigma_{i=0}^{x} f(i) = f(0) + f(1) + \cdots + f(x)$
2. *The function Π:* $\Pi_{i=0}^{x} f(i) = f(0) \times f(1) \times \cdots \times f(x)$

Exercise 2. *Prove Proposition 1.*

All p.r. functions are clearly effectively computable, but:

Theorem 21. *There are effectively computable numerical functions which aren't p.r.*

Proof. Note that the set of p.r. functions is effectively enumerable, i.e. there is an effectively computable injective function mapping each p.r. function to a natural number. Let $f_0, f_1, f_2, \ldots$ be such an enumeration. Consider the following table:

	0	1	2	3	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	...
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	...
...	...	...	...	...	$\ddots$

Let's define the *diagonal function* by putting $\delta(x) = f_x(x)+1$. Assume all effectively computable functions are p.r. Since we are working with an effective enumeration of the p.r. functions, then δ must itself be p.r. Thus, there must be a natural number n s.t. $\delta = f_n$. But then we would have that $f_n(n) = \delta(n) = f_n(n) + 1$, which is impossible. Therefore, not all effectively computable functions are p.r. □

Note that the previous argument also shows that the set of effectively computable *total* functions is not effectively enumerable.[1]

Exercise 3 (*). *Define an effectively computable injective function mapping each p.r. function to a natural number.*

[1] Total functions are those that assign a value to every member of the domain, whilst partial functions might fail to do so.

2 Primitive recursive properties and relations

Definition (characteristic function)**.** *The characteristic function of the numerical property P is the one-place function c_P s.t.*

$$c_P(x) = \begin{cases} 0 & \textit{if } x \textit{ is } P \\ 1 & \textit{if } x \textit{ isn't } P \end{cases}$$

The characteristic function of the n-place numerical relation R is the n-place function c_R s.t.

$$c_R(x_1, \dots, x_n) = \begin{cases} 0 & \textit{if } R(x_1, \dots, x_n) \\ 1 & \textit{otherwise} \end{cases}$$

Definition (p.r. properties and relations)**.** *A p.r. property is a property with a p.r. characteristic function, and likewise a p.r. relation is a relation with a p.r. characteristic function.*

Thus, all p.r. properties and relations are effectively decidable.

The identity relation $=$ is p.r., as its characteristic function can be defined as $c_=(x, y) := sg(|x - y|)$.

Exercise 4. *Show that the following properties and relations are p.r.:*

1. *The property of being even*
2. *The property of being identical to (a given) n*
3. *The inequivalent relation $\neq$*
4. *The equal-to-or-less-than relation $\leq$*
5. *The equal-to-or-less-than relation $<$*
6. *The greater-than relation $>$*
7. *The (3-place) relation of being greater than a number and smaller than the other*

Proposition 2. *If $P(x_1, \dots, x_n)$ is a p.r. (property or) relation and $f(y_1, \dots, y_m)$ is a p.r. function, then $P(x_1, \dots, x_n)[x_i/f(y_1, \dots, y_m)]$ is also a p.r. relation.*

Proof. The characteristic function of $P(x_1, \dots, x_n)[x_i/f(y_1, \dots, y_m)]$ can be defined by composing c_P with f. Since they are both p.r., so is $P(x_1, \dots, x_n)[x_i/f(y_1, \dots, y_m)]$. $\square$

Proposition 3. *Let $P(x_1, \dots, x_n)$ and $Q(x_1, \dots, x_n)$ be p.r. relations. Then, the following are also p.r.:*

1. $\neg P(x_1, \dots, x_n)$
2. $P(x_1, \dots, x_n) \wedge Q(x_1, \dots, x_n)$
3. $P(x_1, \dots, x_n) \vee Q(x_1, \dots, x_n)$
4. $P(x_1, \dots, x_n) \rightarrow Q(x_1, \dots, x_n)$
5. $P(x_1, \dots, x_n) \leftrightarrow Q(x_1, \dots, x_n)$
6. $\forall y \leq x_i \, P(x_1, \dots, x_n)[y/x_i]$
7. $\exists y \leq x_i \, P(x_1, \dots, x_n)[y/x_i]$

Proof. 7. Since $P(x_1, \dots, x_n)$ is p.r., so is $c_P(x_1, \dots x_n)$. Then, the characteristic function of $\exists y \leq x_i \, P(x_1, \dots, x_n)[y/x_i]$ is also p.r., as it can be defined as $\Pi_{j=0}^{x_i} c_P(x_1, \dots x_n)[j/x_i]$. $\square$

Exercise 5. *Prove items 1-6 of Proposition 3.*

By Propositions 2 and 3, the relation $|$ of being a divisor of is p.r., as it can be defined as follows:

$$x|y := \exists z \leq y \, (y = x \times z)$$

Exercise 6. *Prove that the property Prime of being a prime number is p.r.*

Proposition 4. *Let $C_1(x_1, \dots, x_n), \dots, C_m(x_1, \dots, x_n)$ be parwise exclusive and jointly exhaustive p.r. relations – i.e. for every n-tuple $\langle k_1, \dots, k_n \rangle$, it must be the case that $C_i(k_1, \dots, k_n)$ for some i s.t. $1 \leq i \leq m$ and, for any i, j s.t. $i \neq j$, it is not the case that $C_i(k_1, \dots, k_n)$ and $C_j(k_1, \dots, k_n)$ – and let $g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n)$ be p.r.functions. Then the function f defined as follows is p.r. as well:*

$$f(x_1, \dots, x_n) = \begin{cases} g_1(x_1, \dots, x_n) & \text{if } C_1(x_1, \dots, x_n) \\ \dots & \dots \\ g_m(x_1, \dots, x_n) & \text{if } C_m(x_1, \dots, x_n) \end{cases}$$

Exercise 7. *Prove Proposition 4.*

Proposition 5. *If P is a p.r. property, then the bounded minimisation function $(\mu y \leq x)P(y)$ is p.r.:*

$$(\mu y \leq x)P(y) = \begin{cases} \text{the smallest number that is a } P \text{ and is equal to or less than } x & \text{if any} \\ x + 1 & \text{otherwise} \end{cases}$$

Exercise 8. *Prove Proposition 5.*

Exercise 9. *Show that the following functions are p.r.:*

1. *The function π that enumerates prime numbers in their natural order: $\pi(x) = x$-th prime.*
2. *The exponent function exf: given two natural numbers, x and y, it returns the (possibly zero) exponent of the y-th prime in the factorisation of x.*
3. *The length function len: given a natural number, x, it returns the number of distinct prime factors of x.*