Gödel's First Incompleteness Theorem

Unit 5: Arithmetization of Syntax

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1 Gödel numbering

When we are dealing with finite objects such as wffs (finite sequences of symbols taken from an enumerable alphabet, see the Appendix to Unit 1), we can associate them with numerical codes. And then we can use arithmetic to talk about these codes and to deal with the arithmetical properties that – via the coding – 'track' syntactic properties of languages and theories.

First, we associate each *symbol* of L_A with a number, its *basic code*, as follows:

$\neg$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	$=$	(	)	$\mathsf{0}$	S	$+$	$\times$	x	y	z	$\ldots$
1	3	5	7	9	11	13	15	17	19	21	23	25	27	2	4	6	$\ldots$

Note that this coding is *effective*, as there is an algorithm that, given a number, returns the symbol coded by the number and, vice versa, there is an algorithm that, given any symbol returns its basic code.

Definition 1 (Gödel number)**.** *If the expression e of L_A is the sequence of $n+1$ symbols $s_0, s_1, s_2, \ldots, s_n$, then e's Gödel number (g.n.) – $\ulcorner e \urcorner$ – is calculated by taking the basic code c_i for each s_i in turn, using c_i as an exponent for the $i+1$-th prime number $\pi(i)$, and then multiplying the results, to get $2^{c_0} \times 3^{c_1} \times 5^{c_2} \times \cdots \times \pi(n)^{c_n}$.*[1]

For instance, the g.n. of Sz, $\ulcorner \mathsf{Sz} \urcorner$, is $2^{23} \times 3^6$, $\ulcorner \mathsf{x} = \mathsf{S0} \urcorner$ is $2^2 \times 3^{15} \times 5^{23} \times 7^{21}$, and the g.n. of $\neg\exists \mathsf{y}\, \mathsf{y} + \mathsf{0} = \mathsf{0}$ is $2^1 \times 3^{13} \times 5^4 \times 7^4 \times 11^{25} \times 13^{21} \times 17^{15} \times 19^{21}$.

Definition 2 (super Gödel number)**.** *Given a sequence $e_0, e_1, e_2, \ldots, e_n$ of the expressions of L_A, the super Gödel number (s.g.n.) of the sequence is calculated by taking the g.n. g_i for each e_i in turn, using g_i as an exponent for the $i+1$-th prime number $\pi(i)$, and then multiplying the results, to get $2^{g_0} \times 3^{g_1} \times 5^{g_2} \times \cdots \times \pi(n)^{g_n}$.*

For instance, the g.n. of the sequence of wffs $\neg\exists \mathsf{y}\, \mathsf{y} + \mathsf{0} = \mathsf{0}$, $\mathsf{x} = \mathsf{S0}$ is

$$2^{2^2 \times 3^{15} \times 5^{23} \times 7^{21}} \times 3^{2^1 \times 3^{13} \times 5^4 \times 7^4 \times 11^{25} \times 13^{21} \times 17^{15} \times 19^{21}}$$

By the Fundamental Theorem of Arithmetic, each natural number greater than 1 can be written uniquely as a product of prime numbers. Moreover, given any number, there is an *algorithm* that gives us this factorization, and an algorithm that takes us from such a product to its result. Since our basic coding is effective, this means that our Gödel numbering is also *effective*. All the

[1] See Unit 3, Exercise 9.

results that follow depend only on the numbering being effective and not on the particular way it is defined here. Thus, we say that our results are *robust*.

Note that, if e is a subsequence of e', then $\ulcorner e \urcorner \leq \ulcorner e' \urcorner$ and, if a sequence is a subsequence of another sequence of expressions, then the s.g.n of the former cannot be larger than the latter. This means that our Gödel numbering is *monotonic*.

2 Arithmetization of syntactic functions, properties, and relations

Proposition 1. *The numerical property $Var(n)$, that holds when n is the g.n. (not the basic code!) of a variable, is p.r.*

Proof. Let $Var(n) := \exists x \leq n\, (x \neq 0 \wedge n = 2^{2x})$. By Propositions 2 and 3 and Exercise 4 of Unit 3, Var is p.r. □

Proposition 2. *The numerical properties $Exp(n)$, that holds when n is the g.n. of a sequence of symbols, and $Seq(n)$, that holds when n is the s.g.n. of a sequence of sequences of symbols, are both p.r.*

Proof. Let $Exp(n) := n > 1 \wedge \forall x \leq n\, ((exf(n, x) > 0 \rightarrow \forall y < x\; exf(n, y) > 0) \wedge (\neg \exists y\; exf(n, x) = 2y \rightarrow exf(n, x) \leq 27))$. By Propositions 2 and 3 and Exercises 4 and 9 of Unit 3, Exp is p.r. □

Exercise 1. *Complete the proof of Proposition 2 showing that $Seq(n)$ is a p.r. property.*

Proposition 3. *The concatenation function, $\frown$, defined as follows is p.r.:*

- $x \frown y = \begin{cases} \ulcorner e_1 e_2 \urcorner & \text{if } e_1 \text{ and } e_2 \text{ are expressions, and } x = \ulcorner e_1 \urcorner \text{ and } y = \ulcorner e_2 \urcorner \\ 0 & \text{otherwise} \end{cases}$

Given the g.n. of two sequences of symbols, the concatenation function returns the g.n. of their concatenation.

Proof. Let $x \frown y := x \times \Pi^{len(y)-1} \pi(len(x+i))^{exf(y,i)}$. By Proposition 1 and Exercise 9 of Unit 3, $\frown$ is p.r. □

Exercise 2. *Show that the following functions are p.r. (if not fed the appropriate numbers, they return* 0*):*

1. *The negation function, $\dot{\neg}$: $\dot{\neg} x$ is the g.n. of the expression that results from writing a negation symbol in front of the expression with g.n. x.*
2. *The conditional function, $\dot{\rightarrow}$: $x \dot{\rightarrow} y$ is the g.n. of the expression that results from writing a conditional symbol between the expression with g.n. x and the expression with g.n. y and brackets around it.*
3. *The conjunction function, $\dot{\wedge}$: $x \dot{\wedge} y$ is the g.n. of the expression that results from writing a conjunction symbol between the expression with g.n. x and the expression with g.n. y and brackets around it.*

4. *The biconditional function,* $\dot{\leftrightarrow}$*:* $x\dot{\leftrightarrow}y$ *is the g.n. of the expression that results from writing a biconditional symbol between the expression with g.n.* x *and the expression with g.n.* y *and brackets around it.*
5. *The disjunction function,* $\dot{\vee}$*:* $x\dot{\vee}y$ *is the g.n. of the expression that results from writing a disjunction symbol between the expression with g.n.* x *and the expression with g.n.* y *and brackets around it.*
6. *The universal quantifier function,* $\dot{\forall}$*:* $\dot{\forall}xy$ *is the g.n. of the expression that results from writing a universal quantifier followed by the variable with g.n.* x *in front of the expression with g.n.* y*.*
7. *The existential quantifier function,* $\dot{\exists}$*:* $\dot{\exists}xy$ *is the g.n. of the expression that results from writing an existential quantifier followed by the variable with g.n.* x *in front of the expression with g.n.* y*.*

Theorem 28. *The following numerical properties are p.r.:*

1. $Term(n)$*, to hold when* n *is the g.n. of a term of* L_A*.*
2. $Wff(n)$*, to hold when* n *is the g.n. of a wff of* L_A*.*
3. $Sent(n)$*, to hold when* n *is the g.n. of a sentence of* L_A*.*

Proof. We can state $Term(n)$ as follows:

(A_T) There is a sequence of numbers $k_0, k_1, \ldots, k_x$ s.t. for every $y \leq x$, k_y is the g.n. of $\mathsf{0}$ or a variable or there is a $z < y$ s.t. k_y is the g.n. of the result of writing S in front of the expression with g.n. k_z, or there are $z < y$ and $w < y$ s.t. k_y is the g.n. of the result of writing $+$ or $\times$ between the expressions with g.n. k_z and k_w; and $k_x = n$.

Thus, coding sequences of numbers using their prime factorization, we can define $Term(n)$ as follows:

(B_T) $Term(n) := \exists x \leq \pi\, (\forall y < len(x)\, (exf(x, y) = 2^{21} \vee Var(exf(x, y)) \vee$
$\exists z < y\, (exf(x, y) = 2^{23} \frown exf(x, z)) \vee$
$\exists z < y \exists w < y\, (exf(x, y) = exf(x, z) \frown 2^{25} \frown exf(x, w) \vee exf(x, y) = exf(x, z) \frown 2^{27} \frown exf(x, w)))$
$\wedge\ exf(x, len(x) \dot{-} 1) = n)$.

□

Exercise 3. *Show that the identity symbol function* $\dot{=}$*, defined as follows, is p.r.:*

- $$x \dot{=} y = \begin{cases} \ulcorner \tau_1 = \tau_2 \urcorner & \text{if } \tau_1 \text{ and } \tau_2 \text{ are terms of } L_A,\ x = \ulcorner \tau_1 \urcorner \text{ and } y = \ulcorner \tau_2 \urcorner \\ 0 & \text{otherwise} \end{cases}$$

Exercise 4 (*)**.** *Prove item 2 of Theorem 28.*

Exercise 5 (**)**.** *First show that the numerical relation* $Free(m, n)$*, that holds when* m *is the g.n. of a variable that is free in the wff with g.n.* n*, is p.r. and then prove item 3 of Theorem 28.*

Exercise 6. *Show that, for every* i *s.t.* $1 \leq i \leq 12$*, the property* $Ax_i(n)$*, that holds of* n *just in case* n *is the g.n. of an instance of Axiom* i *of the calculus for* L_A*, is p.r.*[2]

[2] See the Appendix to Unit 1.

Exercise 7. *Show that the numerical property $Ind(n)$, that holds of n just in case n is the g.n. of an instance of the Induction Schema, is p.r.*

Notation. If n is the g.n. of an expression e of L_A – so $n = \ulcorner e \urcorner$ – we sometimes write $\overline{\ulcorner e \urcorner}$ instead of $\overline{n}$, as the latter works as a name for e in L_A via our coding scheme.

Theorem 29. *The numerical relation $Prf(m, n)$, that holds when m is the s.g.n. of a proof in* PA *of the sentence with g.n. n, is p.r.*

Proof. For any sentence φ of L_A, can state that the sequence $\varphi_1, \ldots, \varphi_k$ of wffs of L_A is a derivation of φ in PA as follows:[3]

(A_{Prf}) φ_k is φ and, for every i s.t. $1 \leq i \leq k$, φ_i is either an instance of Axiom 1 of the calculus for L_A or … or an instance of Axiom 12 of the calculus for L_A or Axiom 1 of PA or … or Axiom 6 of PA or an instance of the Induction Schema or there are wffs $\psi \rightarrow \varphi_i$ and ψ that occur earlier in the sequence or there is a wff ψ that occurs earlier in the sequence and a variable v s.t. φ_i is $\forall v\, \psi$.

Thus, we can state $Prf(m, n)$ as follows:

(B_{Prf})
$$
\begin{aligned}
& Seq(m) \wedge exf(m, len(m \dot{-} 1)) = n \wedge \\
& \forall x < len(m) \dot{-} 1\, (Ax_1(exf(m, x)) \vee \cdots \vee Ax_{12}(exf(m, x)) \vee \\
& exf(m, x) = \overline{\ulcorner \text{Axiom } 1 \urcorner} \vee \cdots \vee exf(m, x) = \overline{\ulcorner \text{Axiom } 6 \urcorner} \vee Ind(exf(m, x)) \vee \\
& \exists y < x \exists z < x\, (exf(m, y) = exf(m, z) \dot{\rightarrow} exf(m, x)) \vee \\
& \exists y < x \exists z < n\, (Var(z) \wedge exf(m, x) = \dot{\forall} z\, exf(m, y)))
\end{aligned}
$$

□

Exercise 8. *Let T be a theory formulated in L_A. Show that if the numerical property $Ax_T(n)$, that holds of n just in case n is the g.n. of a non-logical axiom of T, is p.r., then $Prf_T(m, n)$, that holds when m is the s.g.n. of a proof in T of the sentence with g.n. n, is also p.r.*

[3] See the Appendix to Unit 1.